Toric Schemes

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Wir müssen wissen,
wir werden wissen.

D. Hilbert

To Barbara

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Preface

This thesis presents foundations for a theory of toric schemes, generalising the theory of toric varieties. No knowledge about the latter is required from – but of course may be helpful to – the reader. The prerequisites are very moderate throughout. Only some basics in the following fields are assumed:

· foundations, including the language of categories ([E], [7]);

· commutative and homological algebra ([A], [AC], [6]);

· topology, including topological vector spaces ([TG], [EVT]);

· algebraic geometry ([ÉGA, I]).

To be more precise, the references from which something is used to build up the theory are given in a *Logical Bibliography.* Besides this an *Additional Bibliography* is provided, giving references that are cited only in motivating, explaining or historical comments.

Some conventions about notation and terminology are given at the beginning, and moreover indices of notation and of terminology are provided at the end. At the beginning of each chapter or section, general hypotheses and abbreviations kept in force during this part of text are given in italics.

There are very few logical dependencies between the first three chapters, whereas the fourth chapter makes heavy use of most that precedes it. Paragraphs in small print are not logically necessary for what follows them (except further paragraphs in small print) and consist mainly of examples and counterexamples.

I am very grateful to Prof. Markus Brodmann for his support during the past years, including hours of interesting and motivating discussions; I highly appreciate the freedom he gave me of doing what I liked most. I also thank Prof. Ngô Việt Trung for refereeing the thesis, Prof. Rodney Y. Sharp for his interest in my work and his support, and my colleagues at the Institute of Mathematics of the University of Zurich during the past years, especially Stefan Fumasoli for teaching me a lot of algebraic geometry, Felix Fontein for answering a question with a custom-made LaTeX-package, and Urs Kollbrunner and Dominic Schuhmacher for a lot of fun at the Cafeteria. A very special thank-you goes to my Häggy for constant encouraging support.

Introduction

L'objet de la Géométrie algébrique [...]
est donc l'étude des schémas.
A. Grothendieck

The theory of *toric varieties,* developed since the 1970's and still bringing forth a lot of works today, obviously deals with toric *varieties.* Strangely enough, to my knowledge there was never published a serious attempt to develop this theory in the framework of *schemes,* that is, a theory of *toric schemes.* The lecture notes by Kempf, Knudsen, Mumford and Saint-Donat on *toroidal embeddings* ([16]), which might be considered as the first appearance of toric varieties, end with a small step in this direction by swapping the algebraically closed base field for a discrete valuation ring. In later sources, most authors went back to algebraically closed fields or even complex numbers (and of course also made use of the additional structures available in the latter case; see for example Oda's book [20]). But with the well-developed theory of schemes at hand – being moreover easier and more natural than the theory of varieties – there is no reason to do so. On the contrary, from a conceptual point of view it seems desirable to have a theory of toric schemes, and the goal of this thesis is to elaborate – or at least lay the foundations for – such a theory.

Besides this general reason to consider toric schemes there are some more specific ones, mostly the treatment of functorial questions about toric varieties. It was a question of this type, namely representability of Hilbert functors, that was at the origin of this work. The definition of Hilbert functor of a toric variety X (see [14]) obviously leads to arbitrary (or at least affine) base changes of X and hence to "toric varieties over arbitrary base rings", that is, toric schemes. Without a theory of toric schemes every approach to the question of representability of this functor in the spirit of [17] is bound to fail, for known proofs of results about toric varieties often do not just carry over to toric schemes, although the results themselves may do. One reason for this is that a lot of proofs for toric varieties are based on Weil divisor techniques while Weil divisors are not necessarily defined on toric schemes. This implies in particular that a theory of toric schemes will provide new proofs for old results about toric varieties, avoiding Weil divisors and hence being probably easier (or at least more combinatorial).

Throughout this work I tried to attack all the problems in great generality and moreover emphasise functoriality of the involved notions. Not only may these principles be helpful in proving theorems, but I am convinced that in addition they help in understanding what is going on. Functoriality may explain more precisely *how* some term depends on other terms, and generality may reveal *whose guilt* it is that some statement is true or not. These principles and the wish for a rigorous exposition without too many prerequisites – also as a basis for future work – are the reason why some parts of the following are surely well-known, known, or at least partially known in one form or the other.

I will now give a chapterwise brief summary, and inform the reader that there will be more detailed information at the beginning of each chapter.

In Chapter I, the algebraic foundations are treated, that is, *functors of algebras and bigebras of monoids.* Moreover, we consider schemes glued from spectra of bigebras of monoids defined by projective systems of monoids, and investigate their geometry. Later on, toric schemes are defined as a special case of this construction.

Chapter II contains the combinatorial ingredients, that is, *cones* and *fans.* Since I know of no satisfying treatment of this we develop it from scratch in an abstract way, avoiding coordinates and using purely topological arguments if possible. As a highlight of the whole thesis I would like to mention the *Completion Theorem;* it shows that every (simplicial) fan has a (simplicial) completion, that is, it can be extended to a (simplicial) fan that covers the ambient space.

Some more algebraic work is done in Chapter III on *graduations.* We consider rings and modules graded by arbitrary groups and their behaviour under functors like coarsening or restriction of degrees. Furthermore, we develop some homological algebra, including local cohomology, of arbitrarily graded modules.

Finally, in Chapter IV everything is put together. First, a fan Σ gives rise to a projective system of monoids, and the techniques from Chapter I yield for every ring R the *toric scheme* $X_\Sigma(R)$. Second, based on Cox's influental article [10] we associate with every fan Σ and every ring R the so-called *Cox ring* and the so-called *Cox scheme* $Y_\Sigma(R)$, together with a canonical morphism $Y_\Sigma(R) \to X_\Sigma(R)$. This morphism is an isomorphism if and only if Σ is not contained in a hyperplane and hence allows to study Cox schemes instead of toric schemes. Third, using our work on graduations from Chapter III we generalise further results from [10] to describe quasicoherent sheaves of modules on Cox schemes in terms of graded modules over the corresponding Cox ring. More precisely, we show how a graded module over the Cox ring gives rise to a quasicoherent sheaf of modules on the Cox scheme $Y_\Sigma(R)$, that every quasicoherent sheaf of modules on $Y_\Sigma(R)$ arises like this, and – if Σ is simplicial – that this correspondence induces a bijection between graded ideals of a certain restriction of the Cox ring that are saturated with respect to the so-called irrelevant ideal, and quasicoherent ideals of the structure sheaf on $Y_\Sigma(R)$. Finally, a toric version of the *Serre-Grothendieck correspondence* gives a relation between sheaf cohomology on Cox schemes and local cohomology over the corresponding Cox ring.

Notations and conventions

(1) In general, we use the terminology of Bourbaki's *Éléments de mathématique* and Grothendieck's *Éléments de géométrie algébrique,* and unexplained terminology or notation is meant to refer to these treatises. Further remarks and reminders on terminology will be given in footnotes, and we provide indices of notations and of terminology.

(2) As a logical framework we use Bourbaki's Theory of Sets ([E]), including the axioms UA and UB concerning universes ([1, I.0; I.11]). Some of the following is tacitly meant relatively to a universe $\mathcal{U}$ containing an infinite set and chosen once and for all. The objects considered are often tacitly supposed to be elements of $\mathcal{U}$. A few remarks on set theoretical questions will be given in small print and indicated by "Concerning set theory, ...".

(3) We define categories accordingly to [7]. The categories considered are often tacitly supposed to be $\mathcal{U}$-categories or even $\mathcal{U}$-small categories. If C is a category, then we denote by C° its dual category and by $\mathrm{Ob}(\mathsf{C})$ the set of objects of C, and for objects A and B of C we denote by $\mathrm{Hom}_{\mathsf{C}}(A, B)$ the set of morphism in C from A to B and by Id_A the identity morphism of A. If C is a category and A is an object of C, then we denote by $\mathsf{C}_{/A}$ the category of objects over A in C, and by $\mathsf{C}^{/A}$ the category of objects under A in C. If C is an Abelian category, then we denote by $\mathsf{Co}(\mathsf{C})$ and $\mathsf{CCo}(\mathsf{C})$ respectively the categories of complexes and cocomplexes in C (see III.4.1.1).

By a functor we always mean a covariant functor. If C and D are categories, then we denote by $\mathsf{Hom}(\mathsf{C}, \mathsf{D})$ the category of functors from C to D. For a strictly positive number k, by a k-functor (or bifunctor in case $k = 2$) we mean a functor whose source is a product of k categories (although in fact every functor has this property). Further notions involving the variance of the arguments of k-functors like "contra-covariant bifunctor" are hoped to be self-explaining.

In diagrams we use arrows of the form $\rightarrowtail$ and $\twoheadrightarrow$ to denote mono- and epimorphisms, and sometimes we denote canonical injections of subobjects by arrows of the form $\hookrightarrow$.

(4) We denote by Ens the category of sets that are elements of the universe $\mathcal{U}$. We denote by $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ the sets of natural numbers, of strictly positive natural numbers, of integers, and of rational numbers.

(5) We denote by Ord the category of ordered sets that are elements of the universe $\mathcal{U}$, with morphisms the increasing maps. If $(E, \leq)$ is a preordered set and $x \in E$, then we set $E_{\geq x} := \{y \in E \mid y \geq x\}$, and we define analogously the sets $E_{\leq x}$, $E_{>x}$ and $E_{<x}$.

(6) By a monoid or a group we always mean a commutative monoid or group, and we denote by Mon and Ab the categories of monoid and groups that are elements of the universe $\mathcal{U}$. Mostly, monoids will be written additively. If M is a monoid, then we call a set furnished with an M-action an M-monomodule, and if M is an element of $\mathcal{U}$, then we denote by $\mathsf{Mod}(M)$ the category of M-monomodules that are elements of $\mathcal{U}$, with morphisms the homomorphisms of sets with M-actions.

(7) By a ring we always mean a commutative ring, and we denote by Ann the category of rings that are elements of the universe $\mathcal{U}$. If R is a ring, then by an R-algebra we always mean a commutative, unital, associative R-algebra, and by an R-bigebra we always mean a commutative, cocommutative, unital, counital, associative, coassociative R-bigebra. If R is an element of $\mathcal{U}$, then we denote by $\mathsf{Alg}(R)$ and $\mathsf{Big}(R)$ the categories of R-algebras and R-bigebras that are elements of $\mathcal{U}$, with morphisms the unital homomorphisms of R-algebras and R-bigebras. If R' is an R-algebra, then by its multiplication and unit we mean the morphisms of R-modules $R' \otimes_R R' \to R'$ and $R \to R'$ respectively defining the structure on R', and similarly for the comultiplication and counit of an R-bigebra.

(8) Let R be a ring. For $S \subseteq R$ we denote by $S^{-1}R$ the ring of fractions of R with denominators in S and by $\eta_S(R) : R \to S^{-1}R$ or, if no confusion can arise, by η_S the canonical morphism of rings. In case $S = \{f\}$ for some $f \in R$, or $S = R \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$, we write R_f and η_f, or $R_\mathfrak{p}$ and $\eta_\mathfrak{p}$, instead of $S^{-1}R$ and η_S (see also III.2.5.4).

We denote by $\mathrm{Nil}(R)$ and $\mathrm{Idem}(R)$ respectively the sets of nilpotent and idempotent elements of R, and by $\mathrm{Min}(R)$ the set of minimal prime ideals of R.

(9) If R is a ring that is an element of the universe $\mathcal{U}$, then we denote by $\mathsf{Mod}(R)$ the category of R-modules that are elements of $\mathcal{U}$, with morphisms the homomorphisms of R-modules. If R is a ring and E and F are R-modules, then we write $\mathrm{Hom}_R(E, F)$ instead of $\mathrm{Hom}_{\mathsf{Mod}(R)}(E, F)$. For an R-module E and a subset $X \subseteq E$ we denote by $\langle X \rangle_R$, and if no confusion can arise by $\langle X \rangle$, the sub-R-module of E generated by X. If R is a topological ring that is an element of $\mathcal{U}$, then we denote by $\mathsf{TopMod}(R)$ the category of topological R-modules that are elements of $\mathcal{U}$.

Concerning homological algebra we use the terminology of δ-functors from [6].

(10) We denote by Sch the category of schemes that are elements of the universe $\mathcal{U}$. If C is a category and S is a scheme, then by abuse of language a morphism $u : F \to G$ of functors or of contravariant functors from C to $\mathsf{Sch}_{/S}$ is called an immersion, an open immersion, or a closed immersion, respectively, if $u(C)$ is an immersion, an open immersion, or a closed immersion for every $C \in \mathrm{Ob}(\mathsf{C})$. Furthermore, by abuse of language a family $(u_i : F_i \to G)_{i\in I}$ of open immersions of functors or of contravariant functors from C to $\mathsf{Sch}_{/S}$ is called an (affine) open covering of G if $(u_i(C)(F_i(C)))_{i\in I}$ is an (affine) open covering of $G(C)$ for every $C \in \mathrm{Ob}(\mathsf{C})$.

If R is a ring and no confusion can arise, we sometimes write R instead of $\mathrm{Spec}(R)$.

(11) If X is a topological space and $A \subseteq X$, then we denote by $\mathrm{in}_X(A)$, $\mathrm{cl}_X(A)$ and $\mathrm{fr}_X(A)$ respectively the interior, the closure and the frontier of A. If no confusion can arise, we write just $\mathrm{in}(A)$, $\mathrm{cl}(A)$ and $\mathrm{fr}(A)$ for these sets.

CHAPTER I

Algebras of monoids

This chapter introduces and investigates the algebraic foundations for the theory of toric schemes: the functors of algebras and bigebras of monoids.

In Section 1 we start with categorical and elementary generalities on monoids. Then, we define the functors of algebras and bigebras of monoids, that is, a bifunctor

$$\bullet[\blacksquare] : \mathsf{Ann} \times \mathsf{Mon} \to \mathsf{Ann},$$

mapping a ring R and a monoid M onto an R-bigebra $R[M]$. By composition with the contravariant functor $\operatorname{Spec} : \mathsf{Ann} \to \mathsf{Sch}$ we get affine schemes (with some additional structure) of the form $\operatorname{Spec}(R[M])$ for a ring R and a monoid M. If we replace the monoid M by a projective system of monoids $\mathbb{M}$ over a preordered set I, then we can – under the hypothesis that $\mathbb{M}$ is so-called *openly immersive* and that I is a *lower semilattice,* – glue the R-schemes $\operatorname{Spec}(R[\mathbb{M}(i)])$ to obtain an R-scheme $X_{\mathbb{M}}(R)$ (with some additional structure), and this construction is still functorial in R. In Chapter IV, toric schemes will be defined as a special case of this construction. As appropriate for a scheme-theoretical approach we have emphasised throughout functoriality and behaviour under base change of the constructions introduced.

In Section 2 we investigate geometric properties of schemes of the above form $X_{\mathbb{M}}(R)$. We look mainly at the following question:

> *Given an openly immersive projective system of monoids* $\mathbb{M}$ *over a lower semilattice, which properties of schemes are respected or reflected by the functor that maps a ring* R *onto the* R*-scheme* $X_{\mathbb{M}}(R)$*?*

This question can often be reduced to the corresponding question about algebras of monoids, that is:

> *Given a monoid* M*, which properties of rings are respected or reflected by the functor that maps a ring* R *onto the* R*-algebra* $R[M]$ *of a monoid* M*?*

Fortunately a lot is known about this question, and we make use of some non-trivial answers taken from Gilmer's treatise [5] on *Commutative semigroup rings.* As is seen there, some of the properties investigated here, e.g. reducedness, or connectedness, may behave not well in general, but they do under further hypotheses on the monoids involved. We do not hesitate to demand these hypotheses as they are fulfilled by the monoids occuring in the application to toric schemes that we have in mind. So, if the monoid M is torsionfree, cancellable and finitely generated, then a lot of elementary properties are respected and reflected between R and $R[M]$, and we can rephrase this roughly by saying that algebras of monoids (and more general, schemes of the form $X_{\mathbb{M}}(R)$) are as nice (or as ugly) as their base rings. But one should take this with care, for besides the properties investigated here there are reasonable properties with a bad behaviour under the functor $\bullet[M]$. As an example we may mention Cohen-Macaulayness of $R[M]$, depending in general on the characteristic of R as shown by Trung and Hoa in [22].

The choice of properties considered here is somewhat arbitrary but includes enough to show that in general there are too much defects to have a theory of Weil divisors on toric schemes, as mentioned in the Introduction.

1. Spectra of bigebras of monoids

1.1. Monoids, comonoids, and monomodules

Let C *be a category.*

We start with some general nonsense about monoids and comonoids in arbitrary categories. Later on, this will be used on one hand to define a "torus action" on toric schemes, and on the other hand to see that this structure is canonical (1.4.13, IV.1.1.2).

(1.1.1) We define the category $\mathsf{Mon}(\mathsf{C})$ of *monoids in* C accordingly to [ÉGA, 0.1.6][1]. Furthermore, we define the category $\mathsf{Com}(\mathsf{C})$ of *comonoids in* C to be the category $\mathsf{Mon}(\mathsf{C}^\circ)$ of monoids in the dual C° of C.

Now, suppose that C has finite coproducts (and in particular an initial object I). For $A \in \mathrm{Ob}(\mathsf{C})$ let i_A denote the unique morphism $I \to A$, and for $A, B \in \mathrm{Ob}(\mathsf{C})$ let σ_{AB} denote the canonical isomorphism $A \amalg B \xrightarrow{\cong} B \amalg A$. Then, we can spell out the definition of a comonoid in C as follows: a comonoid in C is a pair (C, c) consisting of an object C of C and a morphism $c : C \to C \amalg C$ in C such that the diagrams

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "c"'] & C \amalg C \arrow[d, "\mathrm{Id}_C \amalg c"] \\ C \amalg C \arrow[r, "c \amalg \mathrm{Id}_C"] & C \amalg C \amalg C \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "c"'] \arrow[dr, "c"] & \\ C \amalg C \arrow[rr, "\sigma_{CC}"] & & C \amalg C \end{tikzcd}$$

in C commute (that is, c is *coassociative* and *cocommutative*) and that there is a morphism $u : C \to I$ in C (necessarily unique and called *the counit of* (C, c)) such that the diagram

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "\mathrm{Id}_C"'] & C \amalg C \arrow[d, "\mathrm{Id}_C \amalg u"] \\ C & C \amalg I \arrow[l, "(\mathrm{Id}_C, i_C)"'] \end{tikzcd}$$

in C commutes.

[1] see also [4, I.2]

(1.1.2) Example Let R be a ring. Then, the category $\mathsf{Com}(\mathsf{Alg}(R))$ of comonoids in the category $\mathsf{Alg}(R)$ of R-algebras and the category $\mathsf{Big}(R)$ of R-bigebras are canonically isomorphic (see [A, III.11.4]).

(1.1.3) Example Suppose that C has an empty object $\emptyset$, that is, an object $\emptyset$ such that for every $A \in \mathrm{Ob}(\mathsf{C})$ that is not initial there exists no morphism $A \to \emptyset$. Equivalently, $\emptyset$ is initial, and every morphism with target $\emptyset$ is an isomorphism. In particular, it holds $\emptyset = \emptyset \amalg \emptyset$. If C is a comonoid in C, then the counit of C is a morphism $C \to \emptyset$, and hence $C \cong \emptyset$. Thus, up to unique isomorphism there is a unique comonoid in C, namely $(\emptyset, \emptyset)$.

(1.1.4) Suppose that C has finite direct sums[2]. If (C, c) is a comonoid in C, then the counit of C is the zero morphism $C \to 0$. On use of cocommutativity and identifying $C \amalg C$ and $C \times C$, it is seen that $C \xrightarrow{c} C \amalg C$ is the diagonal morphism of C. Conversely, if $C \in \mathrm{Ob}(\mathsf{C})$, then the diagonal morphism of C in C defines a structure of comonoid in C on C. In other words, on every object of C there exists a unique structure of comonoid, called *canonical,* with comultiplication the diagonal morphism and counit the zero morphism. If moreover $u : C \to D$ is a morphism in C, then u is obviously a morphism of comonoids in C with respect to the canonical structures of comonoids on C and D. Therefore, the forgetful functor $\mathsf{Com}(\mathsf{C}) \to \mathsf{C}$ is an isomorphism of categories.

(1.1.5) Example The category $\mathsf{Mon} = \mathsf{Mon}(\mathsf{Ens})$ of monoids has finite direct sums. Hence, the forgetful functor $\mathsf{Com}(\mathsf{Mon}) \to \mathsf{Mon}$ is an isomorphism.

(1.1.6) Example Suppose that C is an additive category. Then, it has finite direct sums, and hence the forgetful functor $\mathsf{Com}(\mathsf{C}) \to \mathsf{C}$ is an isomorphism.

Now, we already will present the main result of this section, the following proposition and its corollary about extending functors to comonoids.

(1.1.7) Proposition *Let* D *be a further category, suppose that* C *and* D *have finite coproducts, and let* $F : \mathsf{C} \to \mathsf{D}$ *be a functor commuting with finite coproducts. Then, there exists a unique functor* $F_{\mathsf{Com}} : \mathsf{Com}(\mathsf{C}) \to \mathsf{Com}(\mathsf{D})$ *such that the diagram of categories*

$$\begin{tikzcd} \mathsf{Com}(\mathsf{C}) \arrow[r, "F_{\mathsf{Com}}"] \arrow[d] & \mathsf{Com}(\mathsf{D}) \arrow[d] \\ \mathsf{C} \arrow[r, "F"] & \mathsf{D}, \end{tikzcd}$$

where the unmarked functors are the forgetful ones, commutes.

[2]That is, C has finite products and finite coproducts, and the canonical morphism $\coprod_{i \in I} A_i \to \prod_{i \in I} A_i$ is an isomorphism for every finite family $(A_i)_{i \in I}$ in $\mathrm{Ob}(\mathsf{C})$; then, C has in particular a zero object.

Proof. For every comonoid A in C, the comultiplication $A \to A \amalg A$ induces a morphism $F(A) \to F(A) \amalg F(A)$ in D that defines a structure of comonoid in D on $F(A)$. Moreover, if $u : A \to B$ is a morphism in $\mathsf{Com}(\mathsf{C})$, then $F(u) : F(A) \to F(B)$ is obviously a morphism in $\mathsf{Com}(\mathsf{D})$ with respect to the structure of comonoid in D on $F(A)$ and $F(B)$ defined as above. □

(1.1.8) If, in the notation of 1.1.7, no confusion can arise, then we denote the functor F_{Com} by abuse of language again by F.

(1.1.9) Corollary *Suppose that* C *has finite direct sums, let* D *be a category with finite coproducts, and let* $F : \mathsf{C} \to \mathsf{D}$ *be a functor commuting with finite coproducts. Then, there is a unique functor* $F_{\mathsf{Com}} : \mathsf{C} \to \mathsf{Com}(\mathsf{D})$ *such that the diagram of categories*

$$\begin{tikzcd} & \mathsf{Com}(\mathsf{D}) \arrow[d] \\ \mathsf{C} \arrow[ur, "F_{\mathsf{Com}}"] \arrow[r, "F"'] & \mathsf{D}, \end{tikzcd}$$

where the unmarked functor is the forgetful one, commutes.

Proof. Clear from 1.1.4 and 1.1.7. □

(1.1.10) If, in the notation of 1.1.7 and 1.1.9, the functor F is contravariant, we can apply the above to D° to lift F uniquely to a contravariant functor $F_{\mathsf{Com}} : \mathsf{Com}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{D})$ or $F_{\mathsf{Com}} : \mathsf{C} \to \mathsf{Mon}(\mathsf{D})$, respectively.

We end this section with some remarks on monomodules in arbitrary categories. Keep in mind that by an M-monomodule we mean a structure of M-action, where M is a monoid.

(1.1.11) Let M be a monoid in C with multiplication m and unit u. Then, we define the category $\mathsf{Mod}(M)(\mathsf{C})$ of *M-monomodules in* C accordingly to [ÉGA, 0.1.6].

Now, suppose that C has finite products (and in particular a terminal object T). For $A \in \mathrm{Ob}(\mathsf{C})$, we denote by t_A the unique morphism $A \to T$. Then, we can spell out the definition of an M-monomodule in C as follows: An M-monomodule in C is a pair (E, e) consisting of an object E of C and a morphism $e : M \times E \to E$ in C such that the diagrams

$$\begin{tikzcd} M \times M \times E \arrow[r, "\mathrm{Id}_M \times e"] \arrow[d, "m \times \mathrm{Id}_E"'] & M \times E \arrow[d, "e"] \\ M \times E \arrow[r, "e"] & E \end{tikzcd} \qquad \begin{tikzcd} & M \times E \arrow[dr, "e"] & \\ E \arrow[ur, "(u \circ t_M, \mathrm{Id}_E)"] \arrow[rr, "\mathrm{Id}_E"] & & E \end{tikzcd}$$

in C commute. Furthermore, if (E, e) and (F, f) are M-monomodules in C, then a morphism of M-monomodules in C from (E, e) to (F, f) is a morphism $h : E \to F$ in C such that $h \circ e = f \circ (\mathrm{Id}_M \times h)$.

(1.1.12) Let $f : M \to N$ be a morphism in $\mathsf{Mon}(\mathsf{C})$. If E is an N-monomodule in C, then we can consider E by means of f as an M-monomodule in C. This gives rise to a faithful functor

$$\mathsf{Mod}(N)(\mathsf{C}) \to \mathsf{Mod}(M)(\mathsf{C}),$$

called *the scalar restriction functor by means of f.* In particular, as every monoid in C is a monomodule in C over itself, we can consider every monoid in C under M as an M-monomodule in C, and this gives rise to a faithful functor

$$\mathsf{Mon}(\mathsf{C})^{/M} \to \mathsf{Mod}(M)(\mathsf{C}),$$

by abuse of language called *the forgetful functor.*

Now, suppose that C has finite products. If we denote the multiplication of N by n, then the operation of the structure of M-monomodule underlying N is given by $n \circ (f \times \mathrm{Id}_N) : M \times N \to N$.

(1.1.13) Concerning set theory, we have to consider 1.1.1. If C is a $\mathcal{U}$-category, then $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Mod}(M)(\mathsf{C})$ for a monoid M in C enjoy the same property as the forgetful functors $\mathsf{Mon}(\mathsf{C}) \to \mathsf{C}$ and $\mathsf{Mod}(M)(\mathsf{C}) \to \mathsf{C}$ are faithful.

Moreover, if C is $\mathcal{U}$-small, then so are $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Mod}(M)(\mathsf{C})$ for a monoid M in C. Indeed, by the above we need only show that both these categories have $\mathcal{U}$-small sets of objects. In order to achieve this it suffices to show that the set of monoids in C or M-monomodules in C, respectively, with underlying object in C a given $A \in \mathrm{Ob}(\mathsf{C})$ is $\mathcal{U}$-small. So, let $A \in \mathrm{Ob}(\mathsf{C})$. The set of structures of monoid or in C, or M-monomodule in C, respectively, on A is a subset of $\mathrm{Hom}_{\mathsf{C}^\wedge_{\mathcal{U}}}(h_A \times h_A, h_A)$, or of $\mathrm{Hom}_{\mathsf{C}^\wedge_{\mathcal{U}}}(h_M \times h_A, h_A)$, respectively, where $\mathsf{C}^\wedge_{\mathcal{U}}$ denotes the category of $\mathcal{U}$-presheaves on C and h_A and h_M denote the images of A and M under the Yoneda embedding $\mathsf{C} \to \mathsf{C}^\wedge_{\mathcal{U}}$. Since C is $\mathcal{U}$-small we know that $\mathsf{C}^\wedge_{\mathcal{U}}$ is a $\mathcal{U}$-category, and hence $\mathrm{Hom}_{\mathsf{C}^\wedge_{\mathcal{U}}}(h_A \times h_A, h_A)$, or $\mathrm{Hom}_{\mathsf{C}^\wedge_{\mathcal{U}}}(h_M \times h_A, h_A)$, respectively, is $\mathcal{U}$-small. The claim follows from this.

Finally, if $\mathsf{C} \in \mathcal{U}$, then in holds $\mathsf{Mon}(\mathsf{C}) \in \mathcal{U}$ and $\mathsf{Mod}(M)(\mathsf{C}) \in \mathcal{U}$ for a monoid M in C. Indeed, keeping the above notations it suffices to show that for every monoid in C, or every M-monomodule in C, respectively, with underlying object $A \in \mathrm{Ob}(\mathsf{C})$ the corresponding morphism $h_A \times h_A \to h_A$, or $h_M \times h_A \to h_A$, respectively, in $\mathsf{C}^\wedge_{\mathcal{U}}$ is an element of $\mathcal{U}$. But as

$$\mathrm{Hom}_{\mathsf{C}^\wedge_{\mathcal{U}}}(h_A \times h_A, h_A) \subseteq \prod_{B \in \mathrm{Ob}(\mathsf{C})} \mathrm{Hom}_{\mathsf{Ens}}((h_A \times h_A)(B), h_A(B)) \in \mathcal{U}$$

by [1, I.11.1 Proposition 6, Corollaire], and similarly in the case of an M-monomodule, this is clear.

Since analogous statements for C° apply, we get immediately analogous statements for $\mathsf{Com}(\mathsf{C})$.

1.2. Generalities on monoids

We collect here some elementary definitions and results about monoids, used throughout the rest of the thesis.

(1.2.1) A monoid M is called *cancellable* if every element of M is cancellable, that is, if $m + k = m + l$ implies $k = l$ for all $m, k, l \in M$.

Cancellability of monoids may be characterised by monoids of differences. As these will moreover play a fundamental role in Chapter IV, we give a short reminder.

(1.2.2) Let M be a monoid, and let $T \subseteq M$ be a subset. We denote by $M - T$ the monoid of differences of M with negatives in T and by $\varepsilon_T(M)$ or, if no confusion can arise, by ε_T the canonical morphism $M \to M - T$ in Mon. If N is a monoid, then the morphisms $M \to N$ in Mon that map T into the set of invertible elements of N are precisely the morphisms $M \to N$ in Mon that factor over ε_T. If T' is the submonoid of M generated by T, then it holds $M - T = M - T'$. If $T = \{t\}$, then by abuse of language we write $M - t$ and ε_t instead of $M - T$ and ε_T, respectively, and then we have $M - t = M - \mathbb{N}_0 t$. The above shows that ε_T is an epimorphism. It is a monomorphism if and only if every element in T is cancellable.

In case $T = M$ the above yields a left adjoint $\mathrm{Diff} : \mathsf{Mon} \to \mathsf{Ab}$ of the canonical injection $\mathsf{Ab} \to \mathsf{Mon}$. The group $\mathrm{Diff}(M)$ is called *the group of differences of M.* The monoid M is cancellable if and only if the canonical morphism $\varepsilon_M : M \to \mathrm{Diff}(M)$ is a monomorphism ([A, I.2.4]).

(1.2.3) A monoid M is called *torsionfree* if $rm = rn$ implies $m = n$ for all $m, n \in M$ and all $r \in \mathbb{N}$. Obviously, if M is cancellable, then it is torsionfree if and only if the group $\mathrm{Diff}(M)$ is torsionfree. A monoid M is called *aperiodic* if the submonoid of M generated by m is infinite for every $m \in M \setminus \{0\}$, and it is called *integrally closed* if it is cancellable and for every $g \in \mathrm{Diff}(M)$ such that there is an $n \in \mathbb{N}$ with $ng \in M$ it holds $g \in M$.

The properties of being aperiodic or integrally closed will not be used often in what follows, but we show that they are shared by those monoids that will occur often.

(1.2.4) Proposition *Let M be a monoid.*

a) If M is torsionfree and cancellable, then M is aperiodic.

b) If M is torsionsfree, cancellable and finitely generated, then M is integrally closed.

Proof. a) Let $m \in M$ be such that the submonoid of M generated by m is finite. Then, there are $r \in \mathbb{N}_0$ and $s \in \mathbb{N}$ such that $rm = (r+s)m$. Hence, as M is cancellable we get $sm = 0$, and as M is torsionfree it follows $m = 0$. Therefore M is aperiodic.

b) As M is torsionfree and finitely generated, $\mathrm{Diff}(M)$ is a free group with a finite basis $E \subseteq M$. Let $g \in \mathrm{Diff}(M)$ and $n \in \mathbb{N}$ be such that $ng \in M$. Then, there are families $(a_e)_{e\in E}$, $(b_e)_{e\in E}$ and $(c_e)_{e\in E}$ in $\mathbb{N}_0$ with $g = \sum_{e\in E}(a_e - b_e)e$ and $ng = \sum_{e\in E} c_e e$. Hence, we have

$$\textstyle\sum_{e\in E} n(a_e - b_e)e = \sum_{e\in E} c_e e.$$

As E is a basis of $\mathrm{Diff}(G)$, we get $n(a_e - b_e) = c_e$ and thus $a_e - b_e \in \mathbb{N}_0$ for every $e \in E$. This implies $g \in M$, and therefore M is integrally closed. □

As with rings, there are notions of ideals and prime ideals of monoids. These may help to exhibit a strong analogy between rings and monoids, only part of which will surface here. We will use the notion of prime ideal of monoids merely as a technical tool in 1.3.16.

(1.2.5) Let M be a monoid. A *monoideal of* M is a sub-M-monomodule of the M-monomodule underlying M, that is, a subset $A \subseteq M$ such that $M + A \subseteq A$. We denote by $\mathbb{I}_M$ the set of monoideals of M, furnished with the ordering induced by $\subseteq$.

A monoideal $A \subseteq M$ is called *prime* if $M \setminus A$ is a submonoid of M. If $A \subseteq M$ is a prime monoideal and $N \subseteq M$ is a submonoid with $A \subseteq N$, then A is a prime monoideal of N.

(1.2.6) Example Let E be a set. We furnish the free monoid $\mathbb{N}_0^{\oplus E}$ over E with the ordering induced by the product ordering on $\mathbb{N}_0^E$. Then, it is clear that a subset $A \subseteq \mathbb{N}_0^{\oplus E}$ is a monoideal of $\mathbb{N}_0^{\oplus E}$ if and only if for every $x \in A$ and every $y \in \mathbb{N}_0^{\oplus E}$ with $x \leq y$ it holds $y \in A$. In particular, every monoideal of $\mathbb{N}_0^{\oplus E}$ is generated by the set of its minimal elements.

Carrying further the analogy between rings and monoids mentioned above we show next that this fits with the notion of Noetherianity.

(1.2.7) A monoid M is called *Noetherian* if the ordered set $\mathbb{I}_M$ of monoideals of M is Noetherian[3]. This holds if and only if every monoideal of M is finitely generated, as is easily seen.

(1.2.8) Proposition *If E is a set, then the free monoid $\mathbb{N}_0^{\oplus E}$ over E is Noetherian if and only if E is finite.*

Proof. If E is infinite, then the monoideal generated by E is obviously not finitely generated and hence $\mathbb{N}_0^{\oplus E}$ is not Noetherian. For the converse it suffices by 1.2.6 to show that for every $n \in \mathbb{N}_0$ every antichain[4] in $\mathbb{N}_0^n$, furnished with the product ordering, is finite. This we do by induction on n.

For $n \leq 1$ it is obvious. So, let $n > 1$, suppose that every antichain in $\mathbb{N}_0^{n-1}$ is finite, and assume that there is an infinite antichain $A \subseteq \mathbb{N}_0^n$. We identify $\mathbb{N}_0^n$ with $\mathbb{N}_0^{n-1} \oplus \mathbb{N}_0$ and denote by p and q respectively the first and the second canonical projection, and we set $B := p(A)$. If $x, y \in A$ are such that $p(x) = p(y)$, then comparability of $q(x)$ and $q(y)$ implies $x = y$. Thus, the restriction of p to A is injective, and therefore B is infinite.

Using the fact that the set of minimal elements of a nonempty subset of $\mathbb{N}_0^{n-1}$ is nonempty and finite, we can recursively define a strictly increasing

[3] An ordered set E is called *Noetherian* if every increasing sequence in E is stationary. This is the case if and only if every nonempty subset of E has a maximal element ([E, III.7.5 Proposition 6]).

[4] To avoid confusion we call a subset of an ordered set E an *antichain (in E)* if it is free (in the terminology of [E]), that is, if its elements are pairwise incomparable.

sequence $(b_i)_{i\in\mathbb{N}_0}$ in B such that $\{x \in B \mid b_i < x\}$ is infinite for every $i \in \mathbb{N}_0$ as follows. First, there exists a minimal element b_0 of B such that $\{x \in B \mid b_0 < x\}$ is infinite. Next, if $i \in \mathbb{N}_0$ and if $b_i \in B$ is such that $\{x \in B \mid b_i < x\}$ is infinite, then there exists a minimal element b_{i+1} of this set such that $\{x \in B \mid b_{i+1} < x\}$ is infinite.

For every $i \in \mathbb{N}_0$ there is a unique preimage $a_i \in A$ of b_i under p. As A is an antichain, the elements a_i and a_{i+1} are incomparable for every $i \in \mathbb{N}_0$, and therefore $p(a_i) < p(a_{i+1})$ implies $q(a_i) > q(a_{i+1})$. But this yields the contradiction that $(q(a_i))_{i\in\mathbb{N}_0}$ is a strictly decreasing sequence in the well-ordered set $\mathbb{N}_0$, and thus the claim is proven. □

A well-known defect of the category Ann of rings is, that a morphism that is a mono- and an epimorphism is not necessarily an isomorphism. In the category Mon of monoids the same defect occurs, and the standard counterexample is essentially the same as in Ann.

(1.2.9) A morphism in Mon is a monomorphism if and only if its underlying map is injective. Indeed, let $f : M \to N$ be a monomorphism in Mon and consider the submonoid $L := \{(x, y) \in M^2 \mid f(x) = f(y)\}$ of M^2. If we denote by p_1 and p_2 the restrictions to L of the canonical projections of M^2, we get $f \circ p_1 = f \circ p_2$, and as f is a monomorphism this implies that $p_1 = p_2$. From this it is easily seen that the map underlying f is injective. The converse is obvious.

A morphism in Mon is an epimorphism if its underlying map is surjective, but the converse does not necessarily hold. Indeed, the first statement is obvious. A counterexample for its converse is given by the canonical injection from the additive monoid $\mathbb{N}_0$ into its group of differences $\mathbb{Z}$ which is an epimorphism by 1.2.2 but obviously not surjective.

(1.2.10) Lemma *Let M be a group, and let $N, P, Q \subseteq M$ be submonoids such that $P \cup (-P) \subseteq Q$. Then, it holds $(N + P) \cap Q = (N \cap Q) + P$.*

Proof. For $x \in N$ and $y \in P$ with $x + y \in Q$ we have $-y \in Q$, hence $x = x + y - y \in Q$ and therefore $x + y \in (N \cap Q) + P$. Conversely, for $x \in N \cap Q$ and $y \in P$ we have $y \in Q$, hence $x + y \in Q$ and therefore $x + y \in (N + P) \cap Q$. □

1.3. Algebras and bigebras of monoids

For a ring R we define the functor $R[\bullet]$ of algebras of monoids over R as adjoint to some forgetful functor, and then we vary R to obtain a bifunctor $\blacksquare[\bullet]$ from $\mathsf{Ann} \times \mathsf{Mon}$ to Ann. By doing so we lose the structure of R-algebra on the rings $R[M]$. Keeping track of this structure amounts to consider $\blacksquare[\bullet]$ as a bifunctor over the first projection of $\mathsf{Ann} \times \mathsf{Mon}$, and this is what we do in this and similar situations.

(1.3.1) Let R be a ring. The forgetful functor $\mathsf{Alg}(R) \to \mathsf{Mon}$, mapping an R-algebra onto its underlying multiplicative monoid, has a left adjoint denoted by

$$R[\bullet] : \mathsf{Mon} \to \mathsf{Alg}(R).$$

If M is a monoid, then the R-algebra $R[M]$ is called *the algebra of M over R* and can be constructed as follows. The R-module underlying $R[M]$ is the free R-module with basis the set underlying M, and hence it is furnished with a map $e : M \to R[M]$ that is injective if and only if $R \neq 0$ or $M = 0$. In this case we transport the structure of monoid of M to the basis $e(M)$ of $R[M]$ and thus get a structure of R-algebra on $R[M]$, and otherwise we furnish $R[M] = 0$ with its unique structure of R-algebra.

In the above notations, we denote by $\mathbb{T}_{R,M} := e(M)$ the canonical basis of $R[M]$ and by $\exp_{R,M} : M \to \mathbb{T}_{R,M}$ the canonical surjection. If no confusion can arise, then we set $e_m := \exp_{R,M}(m)$ for every $m \in M$ and hence write the canonical basis of $R[M]$ as $\{e_m \mid m \in M\}$. Then, the multiplication of $R[M]$ is the morphism $R[M] \otimes_R R[M] \to R[M]$ in $\mathsf{Mod}(R)$ with $e_m \otimes e_n \mapsto e_{m+n}$ for all $m, n \in M$, and the unit of $R[M]$ is the morphism $R \to R[M]$ in $\mathsf{Mod}(R)$ with $1_R \mapsto e_0$.

An element of $\mathbb{T}_{R,M}$ is called a *monomial in $R[M]$*, and a product of an element of R and a monomial in $R[M]$ is called a *term in $R[M]$* ([A, III.2.6]).

(1.3.2) Let M be a monoid. If $h : R \to R'$ is a morphism in Ann, then we consider R' and $R'[M]$ by means of h as R-algebras, and then the canonical morphism $M \to R'[M]$, $m \mapsto e_m$ in Mon induces by 1.3.1 a morphism $h[M] : R[M] \to R'[M]$ in $\mathsf{Alg}(R)$. This gives rise to a functor

$$\bullet[M] : \mathsf{Ann} \to \mathsf{Ann}.$$

Since this maps a ring R onto the R-algebra $R[M]$, we also get a morphism of functors $\mathrm{Id}_{\mathsf{Ann}}(\bullet) \to \bullet[M]$ and thus can consider $\bullet[M]$ as a functor from Ann to Ann under $\mathrm{Id}_{\mathsf{Alg}(R)}$.

If again $h : R \to R'$ is a morphism in Ann, then the induced morphism $h[M] : R[M] \to R'[M]$ in Ann induces by restriction and coastriction a morphism

$$\mathbb{T}_{h,M} : \mathbb{T}_{R,M} \to \mathbb{T}_{R',M},\ e_m \mapsto e_m$$

in Mon such that $\mathbb{T}_{h,M} \circ \exp_{R,M} = \exp_{R',M}$. This is an isomorphism if and only if $R \neq 0$ and $R' \neq 0$, or $M = 0$, and in this case its inverse equals $\exp_{R,M} \circ \exp_{R',M}^{-1}$. The above gives rise to a functor

$$\mathbb{T}_{\bullet,M} : \mathsf{Ann} \to \mathsf{Mon}$$

under the constant functor $\mathsf{Ann} \to \mathsf{Mon}$ with value M, and $\mathbb{T}_{\bullet,M}$ is a subfunctor of the composition of $\bullet[M]$ with the forgetful functor $\mathsf{Ann} \to \mathsf{Mon}$ mapping a ring onto its underlying multiplicative monoid.

(1.3.3) It follows from 1.3.1 and 1.3.2 that the functors $\bullet[M] : \mathsf{Ann} \to \mathsf{Ann}$ under $\mathrm{Id}_{\mathsf{Ann}}$ for varying monoids M give rise to a bifunctor

$$\bullet[\blacksquare] : \mathsf{Ann} \times \mathsf{Mon} \to \mathsf{Ann}$$

under the canonical projection $\mathrm{pr}_1 : \mathsf{Ann} \times \mathsf{Mon} \to \mathsf{Ann}$.

Now, we can apply the results from 1.1 in order to turn our algebras of monoids into bigebras of monoids in a canonical way.

(1.3.4) Let R be a ring. As it has a right adjoint by 1.3.1, the functor $R[\bullet] : \mathsf{Mon} \to \mathsf{Alg}(R)$ commutes with inductive limits by [1, I.2.11], and in particular with coproducts. Hence, by 1.1.9 there is a unique functor $R[\bullet]_{\mathsf{Com}} : \mathsf{Mon} \to \mathsf{Big}(R)$, denoted by abuse of language also by $R[\bullet]$, such that the diagram of categories

$$\begin{tikzcd} & \mathsf{Big}(R) \arrow[d] \\ \mathsf{Mon} \arrow[ur, "R[\bullet]"] \arrow[r, "R[\bullet]"'] & \mathsf{Alg}(R), \end{tikzcd}$$

where the unmarked functor is the forgetful one, commutes.

If M is a monoid, then the R-bigebra $R[M]$ is called *the bigebra of M over R.* Its comultiplication is given by the morphism $R[M] \to R[M] \otimes_R R[M]$ in $\mathsf{Alg}(R)$ with $e_m \mapsto e_m \otimes e_m$ for every $m \in M$, and the counit of $R[M]$ is given by the codiagonal of the free R-module $R[M]$, that is, the morphism $R[M] \to R$ in $\mathsf{Alg}(R)$ with $e_m \mapsto 1$ for every $m \in M$ ([A, III.11.4 Exemple 1]).

The next result provides the basis for most of the base change properties we will prove, including "universality statements" about algebras of monoids (1.3.6) and about toric schemes (IV.1.1.9).

(1.3.5) Proposition *Let $F : \mathsf{Ann} \to \mathsf{Ann}$ be a functor under $\mathrm{Id}_{\mathsf{Ann}}$. There is a canonical isomorphism*

$$\bullet[\blacksquare] \otimes_\bullet F(\bullet) \xrightarrow{\cong} F(\bullet)[\blacksquare]$$

of bifunctors from $\mathsf{Ann} \times \mathsf{Mon}$ to Ann under $F \circ \mathrm{pr}_1$ that yields for every ring R a canonical isomorphism

$$R[\blacksquare] \otimes_R F(R) \xrightarrow{\cong} F(R)[\blacksquare]$$

of functors from Mon to $\mathsf{Big}(F(R))$.

Proof. Let R be a ring and let M be a monoid. The map

$$M \to \mathrm{Hom}_R(F(R), F(R)[M]),\ m \mapsto (x \mapsto xe_m)$$

induces a morphism

$$R[M] \to \mathrm{Hom}_R(F(R), F(R)[M])$$

in $\mathsf{Mod}(R)$ with $e_m \mapsto (x \mapsto xe_m)$ for every $m \in M$ and hence a morphism $\alpha(R,M): R[M] \otimes_R F(R) \to F(R)[M]$ in $\mathsf{Mod}(R)$ with $\alpha(R,M)(e_m \otimes x) = xe_m$ for every $m \in M$ and every $x \in F(R)$. This is a morphism in $\mathsf{Big}(F(R))$, as is readily checked. On the other hand, the map

$$M \to R[M] \otimes_R F(R),\ m \mapsto e_m \otimes 1$$

is a morphism in Mon with target the multiplicative monoid underlying $R[M] \otimes_R F(R)$, and hence it induces a morphism $F(R)[M] \to R[M] \otimes_R F(R)$ in $\mathsf{Alg}(R)$ with $e_m \mapsto e_m \otimes 1$ for every $m \in M$. Clearly, this is inverse to the map underlying $\alpha(R,M)$, and since bijective morphisms in $\mathsf{Big}(F(R))$ are isomorphisms it follows that $\alpha(R,M)$ is an isomorphism in $\mathsf{Big}(F(R))$.

Finally, it is easily seen that $\alpha(R,M)$ is natural in R and M, and this yields the claim. □

(1.3.6) Corollary *Let R be a ring, and let R' be an R-algebra. Then, there is a canonical isomorphism*

$$R[\bullet] \otimes_R R' \xrightarrow{\cong} R'[\bullet]$$

of functors from Mon *to* $\mathsf{Big}(R')$.

Proof. Clear from 1.3.5 with $F(R) = R'$. □

(1.3.7) Corollary *Let R be a ring.*

a) If $\mathfrak{a} \subseteq R$ is an ideal, then there is a canonical isomorphism

$$R[\bullet]/(\mathfrak{a}R[\bullet]) \xrightarrow{\cong} (R/\mathfrak{a})[\bullet]$$

of functors from Mon *to* $\mathsf{Big}(R/\mathfrak{a})$.

b) If $S \subseteq R$ is a subset, then there is a canonical isomorphism

$$S^{-1}(R[\bullet]) \xrightarrow{\cong} (S^{-1}R)[\bullet]$$

of functors from Mon *to* $\mathsf{Big}(S^{-1}R)$.

Proof. Apply 1.3.6 with $R' = R/\mathfrak{a}$ and $R' = S^{-1}R$ respectively. □

(1.3.8) Corollary *For every $k \in \mathbb{N}_0$ there is an isomorphism*

$$\bullet[\bigoplus_{i=1}^{k} \blacksquare_i] \xrightarrow{\cong} \bullet[\blacksquare_1]\cdots[\blacksquare_k]$$

of $(k+1)$-functors from $\mathsf{Ann} \times \mathsf{Mon}^k$ *to* Ann *under* pr_1 *that yields for every ring R a canonical isomorphism*

$$R[\bigoplus_{i=1}^{k} \blacksquare_i] \xrightarrow{\cong} R[\blacksquare_1]\cdots[\blacksquare_k]$$

of k-functors from Mon^k *to* $\mathsf{Big}(R)$.

Proof. Let M be a monoid. Keeping in mind that $\bullet[\blacksquare]$ commutes with coproducts in the second argument by 1.3.1 and applying 1.3.5 with $F(\blacksquare) = \blacksquare[M]$, we get isomorphisms

$$\bullet[\blacksquare \oplus M] \cong \bullet[\blacksquare] \otimes_\bullet \bullet[M] \cong \bullet[M][\blacksquare].$$

As these are clearly natural in M, the claim follows by induction on k. $\square$

(1.3.9) Let $(I, \leq)$ be a preordered set, and let $\mathbb{M} = ((M_i)_{i\in I}, (p_{ij})_{i\leq j})$ be a projective system in Mon over I. For a ring R, we can compose $\mathbb{M}$ with $R[\bullet] : \mathsf{Mon} \to \mathsf{Big}(R)$ to obtain a projective system

$$R[\mathbb{M}] = ((R[M_i])_{i\in I}, (R[p_{ij}])_{i\leq j})$$

in $\mathsf{Big}(R)$ over I, and analogously for $R[\bullet] : \mathsf{Mon} \to \mathsf{Alg}(R)$. For an R-algebra R', it follows from 1.3.6 that there is a canonical isomorphism

$$R[\mathbb{M}] \otimes_R R' \cong R'[\mathbb{M}]$$

of projective systems in $\mathsf{Big}(R')$ (or in $\mathsf{Alg}(R')$).

Furthermore, the bifunctor $\bullet[\blacksquare] : \mathsf{Ann} \times \mathsf{Mon} \to \mathsf{Ann}$ under pr_1 corresponds to a functor from Mon to $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Ann})^{/\mathrm{Id}_{\mathsf{Ann}}}$. Composing this with $\mathbb{M}$ we get a projective system

$$\bullet[\mathbb{M}] = \big((\bullet[M_i])_{i\in I}, (\bullet[p_{ij}])_{i\leq j}\big)$$

of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$ over I.

The following notion of restricted Noetherianity unifies the notions of monomial ideal and graded ideal (see III.3.3.1).

(1.3.10) Let R be a ring, let E be an R-module, and let $\mathbb{L} \subseteq E$ be a subset. A sub-R-module $F \subseteq E$ is called $\mathbb{L}$-*generated* if it has a generating set contained in $\mathbb{L}$, and *finitely* $\mathbb{L}$-*generated* if it has a finite generating set contained in $\mathbb{L}$. The sum of a (finite) family of (finitely) $\mathbb{L}$-generated sub-R-modules of E is (finitely) $\mathbb{L}$-generated again.

Moreover, E is called $\mathbb{L}$-*Noetherian* if the set of all $\mathbb{L}$-generated sub-R-modules of E, ordered by inclusion, is Noetherian. Clearly, E is $\mathbb{L}$-Noetherian if and only if every $\mathbb{L}$-generated sub-R-module of E is finitely $\mathbb{L}$-generated. If E is $\mathbb{L}$-Noetherian, then every sub-R-module of E that contains $\mathbb{L}$ is $\mathbb{L}$-Noetherian, too. If $\mathbb{L} \subseteq R$, then the ring R is called $\mathbb{L}$-*Noetherian* if it is $\mathbb{L}$-Noetherian considered as an R-module.

(1.3.11) Let M be a monoid, and let R be a ring. We denote by $\mathbb{I}_{R,M}$ the set of $\mathbb{T}_{R,M}$-generated ideals of $R[M]$, furnished with the ordering induced by $\subseteq$. The morphism $\exp_{R,M}$ in Mon induces a morphism

$$\mathrm{Exp}_{R,M} : \mathbb{I}_M \to \mathbb{I}_{R,M},\ A \mapsto \langle \exp_{R,M}(A)\rangle_{R[M]}$$

in Ord that is an isomorphism if and only if $R \neq 0$ and $R' \neq 0$, or $M = 0$, and then its inverse is given by $\mathfrak{a} \mapsto \exp_{R,M}^{-1}(\mathfrak{a} \cap \mathbb{T}_{R,M})$.

Now, let $h : R \to R'$ be a morphism in Ann. Then, mapping a $\mathbb{T}_{R,M}$-generated ideal $\mathfrak{a} \subseteq R[M]$ onto the image of the canonical morphism

$$\mathfrak{a} \otimes_{R[M]} R'[M] \to R'[M]$$

in $\mathsf{Mod}(R'[M])$ defines a morphism $\mathbb{I}_{h,M} : \mathbb{I}_{R,M} \to \mathbb{I}_{R',M}$ in Ord with

$$\mathbb{I}_{h,M} \circ \mathrm{Exp}_{R,M} = \mathrm{Exp}_{R',M} .$$

In particular, $\mathbb{I}_{h,M}$ is an isomorphism if and only if $R \neq 0$ and $R' \neq 0$, or $M = 0$. The above gives rise to a functor $\mathbb{I}_{\bullet,M} : \mathsf{Ann} \to \mathsf{Ord}$ under the constant functor $\mathsf{Ann} \to \mathsf{Ord}$ with value $\mathbb{I}_M$.

Finally, let $A \subseteq M$ be a monoideal, and let $h : R \to R'$ be a morphism in Ann. Then, $h[M] : R[M] \to R'[M]$ induces by restriction and coastriction a morphism $\mathrm{Exp}_{R,M}(A) \to \mathrm{Exp}_{R',M}(A)$ in $\mathsf{Mod}(R)$. This gives rise to a functor $\mathrm{Exp}_{\bullet,M}(A) : \mathsf{Ann} \to \mathsf{Ab}$ that is a subfunctor of the composition of $\bullet[M]$ with the forgetful functor $\mathsf{Ann} \to \mathsf{Ab}$ which maps a ring onto its underlying additive group.

(1.3.12) Example Let E be a set, let $M = \mathbb{N}_0^{\oplus E}$ be the free monoid with basis E, and let R be a ring. Then, the R-algebra $R[M]$ is the polynomial algebra in $\mathrm{Card}(E)$ indeterminates over R. If no confusion can arise we write $\mathbb{T}_{R,E} := \mathbb{T}_{R,\mathbb{N}_0^{\oplus E}}$ for the monoid of monomials in $R[M]$. If $h : R \to R'$ is a morphism in Ann and $A \subseteq M$ is a monoideal, then 1.3.11 allows us to identify the $R'[M]$-module $\mathrm{Exp}_{R,M}(A) \otimes_{R[M]} R'[M]$ with its canonical image in $R'[M]$ and in particular consider it as an ideal of $R'[M]$, and then it holds $\mathrm{Exp}_{R,M}(A) \otimes_{R[M]} R'[M] = \mathrm{Exp}_{R',M}(A)$.

(1.3.13) Proposition *Let R be a ring, and let $u : M \to N$ be a morphism in Mon. If $R \neq 0$, then u is an epimorphism or a monomorphism, respectively, if and only if $R[u]$ is so.*

Proof. As $R[\bullet]$ has a right adjoint by 1.3.1 it preserves epimorphisms. Conversely, suppose that $R[u]$ is an epimorphism, and let $v, w : N \to P$ be morphisms in Mon with $v \circ u = w \circ u$. It follows $R[v] \circ R[u] = R[w] \circ R[u]$, and in particular these two morphisms coincide on e_m for every $m \in M$. Hence, since $R \neq 0$ we get $v = w$, and thus u is an epimorphism.

Next, we suppose that u is a monomorphism and show that $\mathrm{Ker}(R[u]) = 0$. Let $(r_m)_{m\in M}$ be a family of finite support in R with $R[u](\sum_{m\in M} r_m e_m) = 0$ and hence $\sum_{m\in M} r_m e_{u(m)} = 0$. As u is injective by 1.2.9, the family $(e_{u(m)})_{m\in M}$ in $R[N]$ is free. Therefore, it holds $r_m = 0$ for every $m \in M$, and from this we see that $\mathrm{Ker}(R[u]) = 0$. Conversely, if $R[u]$ is a monomorphism, then its restriction to the set $\{e_m \mid m \in M\}$ is injective, and hence u is a monomorphism, for $R \neq 0$. $\square$

Next, we consider rings of fractions of algebras of monoids. There is an obvious distinction whether the set of denominators consists only of monomials (1.3.14) or not (1.3.15).

(1.3.14) Let M be a monoid, and let $T \subseteq M$ be a subset. Then, we have morphisms

$$\bullet[\varepsilon_T] : \bullet[M] \to \bullet[M - T]$$

and

$$\eta_{\exp_{\bullet,M}(T)} : \bullet[M] \to \exp_{\bullet,M}(T)^{-1}(\bullet[M])$$

of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$.

If R is a ring, then $R[\varepsilon_T]$ maps $\exp_{R,M}(T)$ into the set of invertible elements of $R[M-T]$, and hence corresponds to a morphism

$$u : \exp_{R,M}(T)^{-1}(R[M]) \to R[M - T]$$

in $\mathsf{Alg}(R[M])$ that is readily seen to be natural in R. On the other hand, the morphism from M to the multiplicative monoid underlying the ring $\exp_{R,M}(T)^{-1}(R[M])$ that maps an element m onto the canonical image of e_m in $\exp_{R,M}(T)^{-1}(R[M])$ maps T into the set of invertible elements of $\exp_{R,M}(T)^{-1}(R[M])$. Therefore, it factors over ε_T and thus induces a morphism $R[M-T] \to \exp_{R,M}(T)^{-1}(R[M])$ in $\mathsf{Alg}(R[M])$ that is easily seen to be inverse to u. Thus, the functors $\bullet[M-T]$ and $\exp_{\bullet,M}(T)^{-1}(\bullet[M])$ from Ann to Ann under $\bullet[M]$ are canonically isomorphic.

(1.3.15) Let R be a ring, let M be a monoid, and let $S \subseteq R[M]$ be a subset. We set $T := S \cap R$, denote by $\eta_S : R[M] \to S^{-1}(R[M])$ and $\eta_T : R \to T^{-1}R$ the canonical morphisms in Ann, and set $U := \eta_T[M](S) \subseteq (T^{-1}R)[M]$. Moreover, we denote by $i : R \to R[M]$, $j : T^{-1}R \to (T^{-1}R)[M]$, and $\eta_U : (T^{-1}R)[M] \to U^{-1}((T^{-1}R)[M])$ the canonical morphisms in Ann.

Then, there are morphisms f, g, h in Ann such that the diagrams

$$\begin{tikzcd}
R \arrow[r, "\eta_T"] \arrow[d, "i"'] & T^{-1}R \arrow[r, "j"] \arrow[d, "f"] & (T^{-1}R)[M] \arrow[d, "\eta_U"] \\
R[M] \arrow[r, "\eta_S"] & S^{-1}(R[M]) & U^{-1}((T^{-1}R)[M]) \arrow[l, "g"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
R \arrow[r, "i"] \arrow[d, "\eta_T"'] & R[M] \arrow[r, "\eta_S"] \arrow[d, "\eta_T[M]"] & S^{-1}(R[M]) \arrow[d, "h"] \\
T^{-1}R \arrow[r, "j"] & (T^{-1}R)[M] \arrow[r, "\eta_U"] & U^{-1}((T^{-1}R)[M])
\end{tikzcd}$$

in Ann commute. It is readily checked that g and h are mutually inverse, and hence the $R[M]$-algebras

$$R[M] \xrightarrow{\eta_S} S^{-1}(R[M])$$

and

$$R[M] \xrightarrow{\eta_U \circ \eta_T[M]} U^{-1}((T^{-1}R)[M])$$

are isomorphic.

We end this section with a construction used in IV.1.3.2 for the investigation of properness of toric schemes.

(1.3.16) Let M be a monoid, let $P \subseteq M$ be a prime monoideal, and let R be a ring. We define a map

$$M \to R[M \setminus P],\ x \mapsto \begin{cases} e_x, & \text{if } x \in M \setminus P; \\ 0, & \text{if } x \in P. \end{cases}$$

Since P is prime it is easy to see that this is a morphism in Mon from M to the multiplicative monoid underlying $R[N]$ and hence corresponds to a morphism

$$\vartheta_{M,P}(R) : R[M] \to R[M \setminus P],\ e_x \mapsto \begin{cases} e_x, & \text{if } x \notin P; \\ 0, & \text{if } x \in P \end{cases}$$

in $\mathsf{Alg}(R)$. Obviously, $\vartheta_{M,P}(R)$ is surjective. Moreover, it is clearly natural in R and hence gives rise to a morphism $\vartheta_{M,P} : \bullet[M] \to \bullet[M \setminus P]$ of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$.

Now, let L be a submonoid of M with $P \subseteq L$. Then, P is a prime monoideal of L, and hence $L \setminus P$ is a submonoid of $M \setminus P$. Moreover, the diagram

$$\begin{tikzcd} \bullet[M] \arrow[r, "\vartheta_{M,P}"] & \bullet[M \setminus P] \\ \bullet[L] \arrow[u, tail] \arrow[r, "\vartheta_{L,P}"] & \bullet[L \setminus P], \arrow[u, tail] \end{tikzcd}$$

of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$, where the unmarked morphisms are induced by the canonical injections of L and $L \setminus P$ into M and $M \setminus P$, commutes.

1.4. Spectra of algebras and bigebras of monoids

In this section we translate the above into geometry by taking spectra. In this way, bigebra structures turn into monoid schemes and monomodule schemes.

(1.4.1) Let R be a ring. The contravariant functor Spec from $\mathsf{Alg}(R)$ to $\mathsf{Sch}_{/R}$ commutes with coproducts. Hence, by 1.1.10 there is a unique contravariant functor $\mathrm{Spec}_{\mathsf{Com}} : \mathsf{Big}(R) \to \mathsf{Mon}(\mathsf{Sch}_{/R})$, by abuse of language also denoted by Spec, such that the diagram of categories

$$\begin{tikzcd} \mathsf{Big}(R) \arrow[r, "\mathrm{Spec}"] \arrow[d] & \mathsf{Mon}(\mathsf{Sch}_{/R}) \arrow[d] \\ \mathsf{Alg}(R) \arrow[r, "\mathrm{Spec}"] & \mathsf{Sch}_{/R}, \end{tikzcd}$$

where the unmarked functors are the forgetful ones, commutes.

If A is an R-bigebra with comultiplication $c : A \to A \otimes_R A$ and counit $e : A \to R$, then the multiplication and the unit of the structure of monoid

R-scheme on $\mathrm{Spec}(A)$ are given by the morphisms of R-schemes

$$\mathrm{Spec}(c) : \mathrm{Spec}(A) \times_R \mathrm{Spec}(A) \to \mathrm{Spec}(A)$$

and

$$\mathrm{Spec}(e) : \mathrm{Spec}(R) \to \mathrm{Spec}(A),$$

respectively.

(1.4.2) Let R be a ring, and let A be an R-bigebra. Then, the contravariant functor

$$\mathrm{Spec} : \mathsf{Big}(R) \to \mathsf{Mon}(\mathsf{Sch}_{/R})$$

from 1.4.1 induces a contravariant functor

$$\mathrm{Spec} : \mathsf{Big}(R)_{/A} \to \mathsf{Mon}(\mathsf{Sch}_{/R})^{/\,\mathrm{Spec}(A)}.$$

Its composition with the forgetful functor

$$\mathsf{Mon}(\mathsf{Sch}_{/R})^{/\,\mathrm{Spec}(A)} \to \mathsf{Mod}(\mathrm{Spec}(A))(\mathsf{Sch}_{/R})$$

(see 1.1.12) yields a contravariant functor

$$\mathsf{Big}(R)_{/A} \to \mathsf{Mod}(\mathrm{Spec}(A))(\mathsf{Sch}_{/R}).$$

If $h : B \to A$ is an R-bigebra over A with comultiplication d, then it is seen from 1.1.11 that the $\mathrm{Spec}(A)$-action of the structure of $\mathrm{Spec}(A)$-monomodule R-scheme on $\mathrm{Spec}(B)$ is given by

$$\mathrm{Spec}((h \otimes \mathrm{Id}_B) \circ d)) : \mathrm{Spec}(A) \times_R \mathrm{Spec}(B) \to \mathrm{Spec}(B).$$

(1.4.3) If R is a ring, then composition of $R[\bullet] : \mathsf{Mon} \to \mathsf{Alg}(R)$ with $\mathrm{Spec} : \mathsf{Alg}(R) \to \mathsf{Sch}_{/R}$ yields a contravariant functor

$$\mathrm{Spec}(R[\bullet]) : \mathsf{Mon} \to \mathsf{Sch}_{/R}.$$

Moreover, if M is a monoid, then composition of $\bullet[M] : \mathsf{Ann} \to \mathsf{Ann}$ under $\mathrm{Id}_{\mathsf{Ann}}$ with $\mathrm{Spec} : \mathsf{Ann} \to \mathsf{Sch}$ yields a contravariant functor

$$\mathrm{Spec}(\bullet[M]) : \mathsf{Ann} \to \mathsf{Sch}$$

over Spec.

Finally, composition of the bifunctor $\bullet[\blacksquare] : \mathsf{Ann} \times \mathsf{Mon} \to \mathsf{Ann}$ under the canonical projection $\mathrm{pr}_1 : \mathsf{Ann} \times \mathsf{Mon} \to \mathsf{Ann}$ with $\mathrm{Spec} : \mathsf{Ann} \to \mathsf{Sch}$ yields a contravariant bifunctor

$$\mathrm{Spec}(\bullet[\blacksquare]) : \mathsf{Ann} \times \mathsf{Mon} \to \mathsf{Sch}$$

over $\mathrm{Spec} \circ \mathrm{pr}_1$.

(1.4.4) Let R be a ring. Then, composition of $R[\bullet] : \mathsf{Mon} \to \mathsf{Big}(R)$ with $\mathrm{Spec} : \mathsf{Big}(R) \to \mathsf{Mon}(\mathsf{Sch}_{/R})$ yields a contravariant functor

$$\mathrm{Spec}(R[\bullet]) : \mathsf{Mon} \to \mathsf{Mon}(\mathsf{Sch}_{/R}).$$

If $u : N \to M$ is a morphism in Mon, then this contravariant functor maps u onto the morphism

$$\mathrm{Spec}(R[u]) : \mathrm{Spec}(R[M]) \to \mathrm{Spec}(R[N])$$

in $\mathsf{Mon}(\mathsf{Sch}_{/R})$ and hence by 1.4.2 defines on $\mathrm{Spec}(R[N])$ a structure of $\mathrm{Spec}(R[M])$-monomodule R-scheme. Thus, for a monoid M the above defines a contravariant functor

$$\mathrm{Spec}(R[\bullet]) : \mathsf{Mon}_{/M} \to \mathsf{Mod}(\mathrm{Spec}(R[M]))(\mathsf{Sch}_{/R}).$$

(1.4.5) Proposition *Let $F : \mathsf{Ann} \to \mathsf{Ann}$ be a functor over $\mathrm{Id}_{\mathsf{Ann}}$. Then, there is a canonical isomorphism*

$$\mathrm{Spec}(\bullet[\blacksquare]) \times_{\mathrm{Spec}(\bullet)} \mathrm{Spec}(F(\bullet)) \xrightarrow{\cong} \mathrm{Spec}(F(\bullet)[\blacksquare])$$

of contravariant bifunctors from $\mathsf{Ann} \times \mathsf{Mon}$ to Sch under $\mathrm{Spec} \circ F \circ \mathrm{pr}_1$ that yields for every ring R a canonical isomorphism

$$\mathrm{Spec}(R[\blacksquare]) \times_R F(R) \xrightarrow{\cong} \mathrm{Spec}(F(R)[\blacksquare])$$

of contravariant functors from Mon to $\mathsf{Mon}(\mathsf{Sch}_{/R})$.

Proof. This follows immediately from 1.3.5 and 1.4.4. □

(1.4.6) Corollary *Let R be a ring, and let R' be an R-algebra. Then, there is a canonical isomorphism*

$$\mathrm{Spec}(R[\bullet]) \times_R R' \xrightarrow{\cong} \mathrm{Spec}(R'[\bullet])$$

of contravariant functors from Mon to $\mathsf{Mon}(\mathsf{Sch}_{/R'})$.

Proof. Clear from 1.4.5 (or from 1.3.6 and 1.4.4). □

(1.4.7) Let I be a preordered set, and let $\mathbb{M} = ((M_i)_{i\in I}, (p_{ij})_{i\leq j})$ be a projective system in Mon over I. For $i \in I$ we set

$$X_{\mathbb{M},i} := \mathrm{Spec}(\bullet[M_i]) : \mathsf{Ann} \to \mathsf{Sch}$$

and denote by $t_{\mathbb{M},i} : X_{\mathbb{M},i}(\bullet) \to \mathrm{Spec}(\bullet)$ the canonical morphism, and for $i, j \in I$ with $i \leq j$ we set

$$\iota_{\mathbb{M},i,j}(\bullet) := \mathrm{Spec}(\bullet[p_{ij}]) : X_{\mathbb{M},i}(\bullet) \to X_{\mathbb{M},j}(\bullet).$$

If R is a ring, then composing the projective system $R[\mathbb{M}]$ in $\mathsf{Big}(R)$ over I with $\mathrm{Spec} : \mathsf{Big}(R) \to \mathsf{Mon}(\mathsf{Sch}_{/R})$ yields an inductive system

$$\mathrm{Spec}(R[\mathbb{M}]) = \big((X_{\mathbb{M},i}(R))_{i\in I}, (\iota_{\mathbb{M},i,j}(R))_{i\leq j}\big)$$

in $\mathsf{Mon}(\mathsf{Sch}_{/R})$ over I, and analogously for $\mathrm{Spec} : \mathsf{Alg}(R) \to \mathsf{Sch}_{/R}$. For an R-algebra R', it follows from 1.3.9 that there is a canonical isomorphism

$$\mathrm{Spec}(R[\mathbb{M}]) \times_R R' \cong \mathrm{Spec}(R'[\mathbb{M}])$$

of inductive systems in $\mathsf{Mon}(\mathsf{Sch}_{/R'})$ (or in $\mathsf{Sch}_{/R'}$).

Furthermore, composing the projective system $\bullet[\mathbb{M}]$ of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$ over I with Spec : $\mathsf{Ann} \rightarrow \mathsf{Sch}$ we get an inductive system

$$\mathrm{Spec}(\bullet[\mathbb{M}]) = \big((X_{\mathbb{M},i}(\bullet))_{i\in I}, (\iota_{\mathbb{M},i,j}(\bullet))_{i\leq j}\big)$$

of contravariant functors from Ann to Sch over Spec over I.

As was said above, our aim is to somehow glue the schemes defined by a projective system $\mathbb{M}$ of monoids. The next task is to make sense of this aim.

(1.4.8) Let R be a ring, let I be a preordered set, and let $\mathbb{M}$ be a projective system in Mon over I. We say that $\mathbb{M}$ is *openly immersive for* R if $\iota_{\mathbb{M},i,j}(R)$ is an open immersion for all $i, j \in I$ with $i \leq j$. If R' is an R-algebra and $\mathbb{M}$ is openly immersive for R, then 1.4.7 implies that it is also openly immersive for R', since open immersions are stable under base change by [ÉGA, I.4.3.6]. We say that $\mathbb{M}$ is *openly immersive* if it is openly immersive for $\mathbb{Z}$, or – equivalently – if $\iota_{\mathbb{M},i,j}$ is an open immersion for all $i, j \in I$ with $i \leq j$.

(1.4.9) Let X be a set, and let $(X_i)_{i\in I}$ be a covering of X. On use of the canonical isomorphism between the category of lower semilattices and morphisms of lower semilattices[5] and the category of idempotent, commutative, associative magmas (see [A, I.1 Exercise 15]) it is easily seen that if $(X_i)_{i\in I}$ is injective, then there exists at most one structure of a lower semilattice on I such that for all $i, j \in I$ it holds $X_i \cap X_j = X_{\inf(i,j)}$. If $(X_i)_{i\in I}$ is injective and there exists such a structure, then $(X_i)_{i\in I}$ is said to have *the intersection property,* and then we consider I as a lower semilattice, furnished with the uniquely determined structure described above. Now, suppose we are given a structure of lower semilattice on I such that $X_i \cap X_j = X_{\inf(i,j)}$ for all $i, j \in I$. If no confusion can arise, then by abuse of language $(X_i)_{i\in I}$ is also said to have *the intersection property,* even if it is not injective.

If C is category and G is a functor or a contravariant functor from C to $\mathsf{Sch}_{/S}$, then by abuse of language an open covering $(u_i : F_i \rightarrow G)_{i\in I}$ of G is said to have *the intersection property* if $(u_i(C)(F_i(C)))_{i\in I}$ has the intersection property for every $C \in \mathrm{Ob}(\mathsf{C})$.

(1.4.10) Let R be a ring, let I be a lower semilattice, and let $\mathbb{M}$ be a projective system in Mon over I that is openly immersive for R. Moreover, let R' be an R-algebra. Then, it follows from 1.4.8 and [ÉGA, I.2.4.1; 0.4.1.7] that the family $(X_{\mathbb{M},i}(R'))_{i\in I}$ of R'-schemes can be glued along $(X_{\mathbb{M},\inf(i,j)}(R'))_{(i,j)\in I^2}$ to obtain an R'-scheme

$$t_{\mathbb{M}}(R') : X_{\mathbb{M}}(R') \rightarrow \mathrm{Spec}(R')$$

[5] An ordered set E is called a *lower semilattice* if $\inf(x, y)$ exists for all $x, y \in E$. If E and F are lower semilattices, then a *morphism of lower semilattices from E to F* is a morphism $f : E \rightarrow F$ in Ord such that $f(\inf(x, y)) = \inf(f(x), f(y))$ for all $x, y \in E$.

together with open immersions $\iota_{\mathbb{M},i}(R') : X_{\mathbb{M},i}(R') \rightarrowtail X_{\mathbb{M}}(R')$ in $\mathsf{Sch}_{/R'}$ for all $i \in I$ such that, identifying $X_{\mathbb{M},i}(R')$ with its image under $\iota_{\mathbb{M},i}(R')$ for every $i \in I$, the family $(X_{\mathbb{M},i}(R'))_{i\in I}$ is an affine open covering of $X_{\mathbb{M}}(R')$ that has the intersection property.

The above gives rise to a contravariant functor

$$X_{\mathbb{M}} : \mathsf{Alg}(R) \to \mathsf{Sch}_{/R}$$

over Spec together with open immersions $\iota_{\mathbb{M},i} : X_{\mathbb{M},i} \to X_{\mathbb{M}}$ of such contravariant functors. If R' is an R-algebra, then it follows from 1.4.7 that the diagram of categories

$$\begin{tikzcd} \mathsf{Alg}(R') \arrow[r, "X_{\mathbb{M}}"] & \mathsf{Sch}_{/R'} \\ \mathsf{Alg}(R) \arrow[u, "\bullet\otimes_R R'"] \arrow[r, "X_{\mathbb{M}}"] & \mathsf{Sch}_{/R} \arrow[u, "\bullet\times_R R'"'] \end{tikzcd}$$

commutes up to canonical isomorphism.

(1.4.11) Example Let R be a ring, let I be a lower semilattice, and let $\mathbb{M}$ be a projective system in Mon over I that is openly immersive. Then, it holds $X_{\mathbb{M}}(R) = \emptyset$ if and only if $I = \emptyset$ or $R = 0$. Indeed, if $X_{\mathbb{M}}(R) = \emptyset$ and $I \neq \emptyset$, then there is an $i \in I$ with $R[M_i] = 0$, hence $R = 0$, or $M_i = 0$ and therefore $R = R[M_i] = 0$. The converse holds obviously.

Besides the underlying schemes we would also like to glue the algebraic structures on the affine pieces of the schemes defined by a projective system $\mathbb{M}$ of monoids. This does not lead to any problems and will be done next.

(1.4.12) Let R be a ring, let Y be a monoid R-scheme, let X be an R-scheme, and let $(X_i)_{i\in I}$ be an open covering of X. Moreover, suppose for every $i \in I$ that X_i is furnished with a structure of Y-monomodule R-scheme that induces for every $j \in I$ a structure of sub-Y-monomodule R-scheme on $X_i \cap X_j$, and that for $i, j \in I$ the structures on $X_i \cap X_j$ induces by X_i and X_j coincide. Then, there is unique structure of Y-monomodule R-scheme on X that induces for every $i \in I$ on X_i the given structure of Y-monomodule R-scheme. Indeed, for every $i \in I$ the Y-action on X_i induces by composition a morphism $Y \times X_i \to X$ in $\mathsf{Sch}_{/R}$, and since $(Y \times X_i)_{i\in I}$ is an open covering of $Y \times X$ these morphisms define a unique morphism $Y \times X \to X$ with the desired properties.

(1.4.13) Let R be a ring, let I be a lower semilattice with a smallest element ω, and let $\mathbb{M}$ be a projective system in Mon over I that is openly immersive for R. Moreover, let R' be an R-algebra. Then, $\mathrm{Spec}(R'[\mathbb{M}])$ is an inductive system in $\mathsf{Mon}(\mathsf{Sch}_{/R'})^{/X_{\mathbb{M},\omega}(R)}$over I, and hence composition with the forgetful functor

$$\mathsf{Mon}(\mathsf{Sch}_{/R'})^{/X_{\mathbb{M},\omega}(R')} \to \mathsf{Mod}(X_{\mathbb{M}}(R'))(\mathsf{Sch}_{/R'})$$

(1.4.2) yields an inductive system in $\mathsf{Mod}(X_{\mathbb{M}}(R'))(\mathsf{Sch}_{/R'})$ over I. Hence, keeping in mind 1.4.10 we can apply 1.4.12 to obtain a structure of $X_{\mathbb{M},\omega}(R')$-monomodule R'-scheme on $X_{\mathbb{M}}(R')$ that induces for every $i \in I$ the structure of $X_{\mathbb{M},\omega}(R')$-monomodule R'-scheme on $X_{\mathbb{M},i}(R')$ obtained by applying the forgetful functor to its structure of monoid R'-scheme under $X_{\mathbb{M},\omega}(R')$.

If R'' is an R'-algebra, then we can identify the monoid R''-schemes $X_{\mathbb{M},\omega}(R') \times_{R'} R''$ and $X_{\mathbb{M},\omega}(R'')$ by 1.4.6, and doing this it is easily seen by 1.4.10 that there is a canonical isomorphism

$$X_{\mathbb{M}}(R') \times_{R'} R'' \cong X_{\mathbb{M}}(R'')$$

of $X_{\mathbb{M},\omega}(R'')$-monomodule R''-schemes.

In order to illustrate the above we describe a certain type of openly immersive projective system of monoids. Below, instances of this type will give rise to toric schemes.

(1.4.14) Proposition *Let M be a monoid, and let $t \in M$. Then,*

$$\mathrm{Spec}(\bullet[\varepsilon_t]) : \mathrm{Spec}(\bullet[M - t]) \to \mathrm{Spec}(\bullet[M])$$

is an open immersion of contravariant functors from Ann *to* Sch.

Proof. Let R be a ring. The $R[M]$-algebras $R[\varepsilon_t] : R[M] \to R[M - t]$ and $\eta_{e_t} : R[M] \to R[N]_{e_t}$ are canonically isomorphic by 1.3.14, and thus the claim follows from [ÉGA, I.1.6.6]. □

(1.4.15) Example Let I be a lower semilattice with a smallest element ω, and let $\mathbb{M} = ((M_i)_{i \in I}, (p_{ij})_{i \leq j})$ be a projective system in Mon over I such that for all $i, j \in I$ with $i \leq j$ there is a $t_{ij} \in M_j$ such that $M_i = M_j - t_{ij}$ and $p_{ij} = \varepsilon_{t_{ij}}$. Then, by 1.4.14 it is seen that $\mathbb{M}$ is openly immersive and hence gives rise to a functor $X_{\mathbb{M}} : \mathsf{Ann} \to \mathsf{Sch}$ over Spec that maps a ring R onto the $X_{\mathbb{M},\omega}(R)$-monomodule R-scheme $X_{\mathbb{M},\omega}(R)$.

(1.4.16) Concerning set theory, we have to consider 1.4.10. So, suppose that I is $\mathcal{U}$-small, that $\mathbb{M} = (M_i)_{i \in I}$ takes values in the category Mon of monoids that are elements of $\mathcal{U}$ (or only that the values of $\mathbb{M}$ are monoids with a $\mathcal{U}$-small underlying set), and that $R' \in \mathcal{U}$ (or only that the underlying set of R' is $\mathcal{U}$-small). Then, it holds $X_{\mathbb{M}}(R') \in \mathcal{U}$. Indeed, we have to show that the underlying set of $X_{\mathbb{M}}(R')$ and the sets underlying the values of the structure sheaf of $X_{\mathbb{M}}(R')$ are elements of $\mathcal{U}$. The second claim follows immediately from [1, I.11.2] and [ÉGA, 0.3.3.1; 0.3.2.1], and since our hypotheses imply $R'[M_i] \in \mathcal{U}$ and hence $X_{\mathbb{M},i}(R') \in \mathcal{U}$ for every $i \in I$, the first claim follows from [1, I.11.1 Proposition 1, Corollaire; Proposition 4, Corollaire 2].

2. Geometric properties of algebras of monoids

Let R be a ring.

2.1. Separatedness and finiteness conditions

We start describing the geometry of spectra of algebras of monoids, or more generally of schemes of the form $X_{\mathbb{M}}(R)$, by looking at the basic properties of quasiseparatedness, separatedness, and quasicompactness. The schemes $X_{\mathbb{M}}(R)$ are always quasiseparated, but not necessarily separated. We characterise separatedness independently of R and only in terms of the projective system of monoids $\mathbb{M}$.

(2.1.1) Lemma *Let X be a scheme, and let $(X_i)_{i\in I}$ be an affine open covering of X. Then, it holds:*

a) If $(X_i)_{i\in I}$ has the intersection property, then X is quasiseparated.

b) If $(X_i)_{i\in I}$ has the intersection property, then X is separated if and only if for all $i,j\in I$ the ring $\mathscr{O}_X(X_{\inf(i,j)})$ is generated by the union of the canonical images of $\mathscr{O}_X(X_i)$ and $\mathscr{O}_X(X_j)$.

c) If I is finite, then X is quasicompact.

Proof. a) holds by [ÉGA, I.6.1.2], b) follows from [ÉGA, I.5.3.6], and c) is obvious. □

(2.1.2) Proposition *Let $\mathbb{M}=((M_i)_{i\in I},(p_{ij})_{i\leq j})$ be a projective system in* Mon *over a lower semilattice I, and suppose that $\mathbb{M}$ is openly immersive for R.*

a) $X_{\mathbb{M}}(R)$ is quasiseparated.

b) If $R\neq 0$, then $X_{\mathbb{M}}(R)$ is separated if and only if for all $i,j\in I$ the monoid $M_{\inf(i,j)}$ is generated by $p_{\inf(i,j)i}(M_i)\cup p_{\inf(i,j)j}(M_j)$.

c) If I is finite, then $X_{\mathbb{M}}(R)$ is quasicompact.

Proof. a) and c) hold by 2.1.1 a), c). b) The characterisation of separatedness in 2.1.1 b) is equivalent to the R-algebra $R[M_{\inf(i,j)}]$ being generated by the union of the images of $R[M_i]$ and $R[M_j]$ under $R[p_{\inf(i,j)i}]$ and $R[p_{\inf(i,j)j}]$ respectively for all $i,j\in I$, and this clearly is equivalent to the condition given above. □

Now, we turn to the properties of being (locally) of finite presentation or of finite type. In the next proof the base change result 1.4.6 demonstrates its usefulness for a first time in allowing to reduce to a Noetherian base ring and thus avoiding to introduce some additional coherence hypothesis.

(2.1.3) Proposition *Let I be a lower semilattice, and let $\mathbb{M}=(M_i)_{i\in I}$ be a projective system in* Mon *over I that is openly immersive. Consider the following statements:*

(1) M_i is finitely generated for every $i\in I$, or $R=0$;

(2) $X_{\mathbb{M}}(R)$ is locally of finite presentation over R;

(3) $X_{\mathbb{M}}(R)$ *is locally of finite type over* R*;*

(4) $X_{\mathbb{M}}(R)$ *is of finite presentation over* R*;*

(5) $X_{\mathbb{M}}(R)$ *is of finite type over* R*.*

a) Statements (1)–(3) are equivalent.

b) If I *is finite, then statements (1)–(5) are equivalent.*

Proof. a) Let $i \in I$, and suppose that M_i is finitely generated. Then, the $\mathbb{Z}$-algebra $\mathbb{Z}[M_i]$ is finitely generated, and hence $X_{\mathbb{M},i}(\mathbb{Z})$ is locally of finite type over $\mathbb{Z}$ by [ÉGA, I.6.2.5]. Noetherianity of $\mathbb{Z}$ and [ÉGA, I.6.2.1.2] imply that $X_{\mathbb{M},i}(\mathbb{Z})$ is locally of finite presentation over $\mathbb{Z}$. Since being locally of finite presentation is stable under base change by [ÉGA, I.6.2.6], it follows from 1.4.6 that $X_{\mathbb{M},i}(R)$ is locally of finite presentation over R. This shows that (1) implies (2).

Obviously, (2) implies (3). Now, suppose that $X_{\mathbb{M}}(R)$ is locally of finite type over R and that $R \neq 0$, and let $i \in I$. Then, the R-algebra $R[M_i]$ is finitely generated by [ÉGA, I.6.2.5] and hence has a finite generating set $E \subseteq \mathbb{T}_{R,M}$. Hence, the morphism $R[\mathbb{N}_0^{\oplus E}] \to R[M_i]$ in $\mathsf{Alg}(R)$ with $e \mapsto e$ is surjective and induces by restriction and coastriction a surjective morphism $\mathbb{N}_0^{\oplus E} \to M_i$ in Mon. Therefore, M_i is finitely generated, and hence (1) holds.

b) is clear from a) and 2.1.2 a), c). □

(2.1.4) Corollary *Let* M *be a monoid. Then, the following statements are equivalent:*

(i) $\mathrm{Spec}(R[M])$ *is of finite presentation over* R*;*

(ii) $\mathrm{Spec}(R[M])$ *is of finite type over* R*;*

(iii) M *is finitely generated, or* $R = 0$*.*

Proof. Clear from 2.1.3 b). □

2.2. Faithful flatness

In this section we show that also faithful flatness of $X_{\mathbb{M}}(R)$ is independent of the base ring R and the monoids in the system $\mathbb{M}$.

(2.2.1) Proposition *Let* M *be a monoid. Then,* $\mathrm{Spec}(R[M])$ *is faithfully flat over* R*.*

Proof. The R-module $R[M]$ is free and hence faithfully flat by [AC, I.3.1 Exemple 2], and therefore $\mathrm{Spec}(R[M])$ is faithfully flat over R by [ÉGA, IV.2.2.3]. □

(2.2.2) Corollary *Let* I *be a lower semilattice, and let* $\mathbb{M}$ *be a projective system in* Mon *over* I *that is openly immersive for* R*. Then,* $X_{\mathbb{M}}(R)$ *is faithfully flat over* R *if and only if* $I \neq \emptyset$ *or* $R = 0$*.*

Proof. Clear from 2.2.1 and 1.4.11. □

(2.2.3) Corollary *Let M be a monoid, and let $\mathfrak{a} \subseteq R$ be an ideal. Then, it holds $\mathfrak{a}R[M] \cap R = \mathfrak{a}$.*

PROOF. Clear from 2.2.1 and [AC, I.3.5 Proposition 8]. □

2.3. Reducedness, irreducibility, and integrality

With respect to reducedness, irreducibility and integrality we first treat spectra of algebras of monoids. To do this we rely on results from Gilmer's book [5], and as mentioned above we have to suppose additional hypotheses on the monoids, namely torsionfreeness and cancellability.

(2.3.1) Lemma *Let M be a torsionfree monoid. Then, it holds*

$$\mathrm{Nil}(R[M]) = \mathrm{Nil}(R)R[M].$$

PROOF. This holds by [5, Theorem 9.9]. □

(2.3.2) Proposition *Let M be a torsionfree monoid. Then, there is a canonical isomorphism*

$$\bullet[M]_{\mathrm{red}} \cong (\bullet_{\mathrm{red}})[M]$$

of functors from Ann *to* Ann.

PROOF. On use of 2.3.1 and 1.3.5 (with $F(\bullet) = \bullet_{\mathrm{red}}$) we get canonical isomorphisms

$$\bullet[M]_{\mathrm{red}} \cong \bullet[M] \otimes_{\bullet} (\bullet_{\mathrm{red}}) \cong \bullet_{\mathrm{red}}[M].$$ □

(2.3.3) Proposition *Let M be a monoid.*

a) If M is torsionfree, then $R[M]$ is reduced if and only if R is reduced.

b) If M is torsionfree and cancellable, then $R[M]$ is integral if and only if R is integral.

PROOF. a) is clear from 2.3.2, and b) holds by [5, Theorem 8.1]. □

(2.3.4) Corollary *Let M be a torsionfree, cancellable monoid, and let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then, it holds $\mathfrak{p}R[M] \in \mathrm{Spec}(R[M])$.*

PROOF. It follows from 1.3.7 a) that $R[M]/\mathfrak{p}R[M] \cong (R/\mathfrak{p})[M]$, and as this ring is integral by 2.3.3 b) the claim is proven. □

(2.3.5) Corollary *Let M be a torsionfree, cancellable monoid. Then*[6]*, $R[M]$ is irreducible if and only if R is irreducible.*

PROOF. If $R[M]$ is irreducible, then so is R by [AC, II.4.1 Proposition 4], since the canonical morphism $\mathrm{Spec}(R[M]) \to \mathrm{Spec}(R)$ is faithfully flat by 2.2.1 and in particular surjective. Conversely, suppose that R is irreducible. Then, it follows from [ÉGA, I.4.5.4] that R_{red} is integral. Hence, $R[M]_{\mathrm{red}}$ is integral by 2.3.2, and another application of [ÉGA, I.4.5.4] shows that $R[M]$ is irreducible. □

[6] By abuse of language we call a ring *irreducible* if its spectrum is irreducible.

To apply the above to schemes of the form $X_{\mathbb{M}}(R)$ we need to know when irreducibility is preserved under glueing. An answer is provided by the following general topological result.

(2.3.6) Proposition *Let X be a topological space, let $(X_i)_{i\in I}$ be an open covering of X that has the intersection property, and suppose that there is a set K such that the following statements hold:*

i) For every $i \in I$, the family[7] of irreducible components of X_i has the form $(Z_{ik})_{k\in K}$;

ii) For every $k \in K$, the covering $(Z_{ik})_{i\in I}$ of $\bigcup_{i\in I} Z_{ik}$ has the intersection property;

iii) For all $i, j \in I$ with $i \leq j$ and all $k \in K$, the set Z_{ik} is an open subset of Z_{jk}.

Then, $(\bigcup_{i\in I} Z_{ik})_{k\in K}$ is the family of irreducible components of X.

Proof. For $k \in K$ we set $Z_k := \bigcup_{i\in I} Z_{ik}$. Obviously, $(Z_k)_{k\in K}$ is a covering of X. If $I = \emptyset$, then the claim is obviously true. So, let $I \neq \emptyset$, and let $k \in K$. First, we show that Z_k is irreducible. As $I \neq \emptyset$, we have $Z_k \neq \emptyset$. Let $U, V \subseteq Z_k$ be nonempty open subsets. We have to show that $U \cap V \neq \emptyset$. There are $i, j \in I$ such that $U \cap Z_{ik}$ and $V \cap Z_{jk}$ respectively are nonempty open subsets of Z_{ik} and Z_{jk}. We set $l := \inf(i, j)$. By ii) and iii), $U \cap Z_{lk}$ and $V \cap Z_{lk}$ are nonempty open subsets of the irreducible set Z_{lk}, and this implies

$$\emptyset \neq U \cap V \cap Z_{lk} \subseteq U \cap V.$$

Thus, Z_k is irreducible.

Now, let $Y \subseteq X$ be an irreducible, closed subset, and suppose that $Z_k \subseteq Y$. Then, for every $i \in I$ it holds $Z_{ik} \subseteq Y \cap X_i$, and $Y \cap X_i$ is an irreducible closed subset of X_i by [AC, II.4.1 Proposition 7]. This implies that $Z_{ik} = Y \cap X_i$ for every $i \in I$, and from this we get $Y = \bigcup_{i\in I} Y \cap X_i = \bigcup_{i\in I} Z_{ik} = Z_k$. Thus, Z_k is an irreducible component of X.

Finally, let $k, l \in K$, and suppose that $Z_k = Z_l$. Let $i \in I$. Since $\emptyset \neq Z_{ik} \subseteq Z_l \cap X_i$, it follows from [AC, II.4.1 Proposition 7] that $Z_l \cap X_i$ is an irreducible component of X_i and hence equal to Z_{ik}. Moreover, as $Z_{il} \subseteq Z_l \cap X_i = Z_{ik}$, we get $Z_{ik} = Z_{il}$ and hence $k = l$. Herewith, the claim is proven. □

(2.3.7) Corollary *Let X be a topological space, and let $(X_i)_{i\in I}$ be an open covering of X with the intersection property such that $X_i \neq \emptyset$ for every $i \in I$. Then, X is irreducible if and only if X_i is irreducible for every $i \in I$ and $I \neq \emptyset$.*

Proof. This follows immediately from 2.3.6. □

[7] By definition, the family of irreducible components of a topological space is injective.

(2.3.8) Corollary *Let I be a lower semilattice, and let $\mathbb{M} = (M_i)_{i\in I}$ be a projective system in* Mon *over I that is openly immersive for R.*

a) If M_i is torsionfree for every $i \in I$, then $X_{\mathbb{M}}(R)$ is reduced if and only if R is reduced or $I = \emptyset$.

b) If M_i is torsionfree and cancellable for every $i \in I$, then $X_{\mathbb{M}}(R)$ is irreducible if and only if R is irreducible and $I \neq \emptyset$.

c) If M_i is torsionfree and cancellable for every $i \in I$, then $X_{\mathbb{M}}(R)$ is integral if and only if R is integral and $I \neq \emptyset$.

Proof. a) follows from 2.3.3 a) and [ÉGA, I.4.5.4], b) is clear from 2.3.7 and 2.3.5, and c) follows from a), b) and [ÉGA, I.4.5.4]. □

(2.3.9) On use of further results in [5, Chapter 9] it can be shown that torsionfreeness of the monoids involved is necessary for 2.3.1, 2.3.2, 2.3.3 a) and 2.3.8 a) to hold independently of the ring R. Similarly, from [5, Theorem 8.1] it can be seen that torsionfreeness and cancellability of the monoids involved are necessary for 2.3.3 b), 2.3.4, 2.3.5 and the rest of 2.3.8 to hold independently of the ring R.

After having seen that reducedness, irreducibility and integrality behave well under the functors $X_{\mathbb{M}}$ we will go on and show that also the notion of decomposition into irreducible components does this.

(2.3.10) Proposition *Let M be a torsionfree, cancellable monoid. Then, the map $\mathfrak{p} \mapsto \mathfrak{p}R[M]$ defines a bijection from* $\mathrm{Min}(R)$ *onto* $\mathrm{Min}(R[M])$*, and its inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cap R$.*

Proof. Let $\mathfrak{p} \in \mathrm{Min}(R)$. Then, we have $\mathfrak{p}R[M] \in \mathrm{Spec}(R[M])$ by 2.3.4 and $\mathfrak{p}R[M] \cap R = \mathfrak{p}$ by 2.2.3. Let $\mathfrak{q} \in \mathrm{Spec}(R[M])$ with $\mathfrak{q} \subseteq \mathfrak{p}R[M]$. Then, it holds $\mathfrak{q} \cap R \subseteq \mathfrak{p}R[M] \cap R = \mathfrak{p}$, therefore $\mathfrak{q} \cap R = \mathfrak{p}$, and thus

$$\mathfrak{p}R[M] = (\mathfrak{q} \cap R)R[M] \subseteq \mathfrak{q} \subseteq \mathfrak{p}R[M].$$

This shows that $\mathfrak{p}R[M] \in \mathrm{Min}(R[M])$.

Conversely, let $\mathfrak{p} \in \mathrm{Min}(R[M])$. Then, it holds $\mathfrak{p} \cap R \in \mathrm{Spec}(R)$, and there exists a $\mathfrak{q} \in \mathrm{Min}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p} \cap R$ by [AC, II.2.6 Lemme 2]. It holds $\mathfrak{q}R[M] \cap R \subseteq (\mathfrak{p} \cap R)R[M] \subseteq \mathfrak{p}$, and 2.3.4 implies $\mathfrak{q}R[M] = (\mathfrak{p} \cap R)R[M] = \mathfrak{p}$. Therefore, we have $\mathfrak{p} \cap R = \mathfrak{q}R[M] \cap R = \mathfrak{q} \in \mathrm{Min}(R)$ by 2.2.3. Herewith the claim is shown. □

(2.3.11) Corollary *Let M be a torsionfree, cancellable monoid. Then, the map $\mathfrak{p} \mapsto$* $\mathrm{Spec}(R/\mathfrak{p}[M])$ *defines a bijection from* $\mathrm{Min}(R)$ *onto the set of irreducible components of* $\mathrm{Spec}(R[M])$.

Proof. By [AC, II.4.3 Proposition 14, Corollaire 2], the map given by $\mathfrak{p} \mapsto \mathrm{Spec}(R[M]/\mathfrak{p})$ defines a bijection from $\mathrm{Min}(R[M])$ onto the set of irreducible components of $\mathrm{Spec}(R[M])$. The claim follows now from 2.3.10 and 1.3.7 a). □

(2.3.12) Corollary *Let I be a lower semilattice, let $\mathbb{M} = (M_i)_{i\in I}$ be a projective system in* Mon *over I that is openly immersive for R, and suppose that $I \neq \emptyset$ and that M_i is torsionfree and cancellable for every $i \in I$. Then, the map $\mathfrak{p} \mapsto X_{\mathbb{M}}(R/\mathfrak{p})$ defines a bijection from $\mathrm{Min}(R)$ onto the set of irreducible components of $X_{\mathbb{M}}(R)$.*

Proof. Clear from 2.3.11 and 2.3.6. □

(2.3.13) A statement analogous to 2.3.9 can be made about 2.3.10 and its corollaries.

2.4. Connectedness

Similar as with irreducibility, our treatment of connectedness comes down to a result from [5] for the affine case together with a general topological result handling the glueing. We also have to restrict to torsionfree and cancellable monoids.

(2.4.1) Lemma *Let X be a topological space, and let $(X_i)_{i\in I}$ be a covering of X that has the intersection property. Suppose that I has a smallest element and that X_i is nonempty and connected for every $i \in I$. Then, X is connected.*

Proof. The hypotheses imply $\bigcap_{i\in I} X_i \neq \emptyset$, and hence the claim follows from [TG, I.11.1 Proposition 2]. □

(2.4.2) Proposition *Let M be a torsionfree, cancellable monoid. Then*[8]*, $R[M]$ is connected if and only if R is connected.*

Proof. By [AC, II.4.3 Proposition 15, Corollaire 2], the affine schemes $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R[M])$ respectively are connected if and only if $\mathrm{Idem}(R) = \{0, 1\}$ and $\mathrm{Idem}(R[M]) = \{0, 1\}$. But as [5, Corollary 10.8] together with 1.2.4 a) implies $\mathrm{Idem}(R) = \mathrm{Idem}(R[M])$, the claim is proven. □

(2.4.3) Corollary *Let I be a lower semilattice with a smallest element, let $\mathbb{M} = (M_i)_{i\in I}$ be a projective system in* Mon *over I that is openly immersive for R, and suppose that M_i is torsionfree and cancellable for every $i \in I$. Then, $X_{\mathbb{M}}(R)$ is connected if and only if R is connected, or $I = \emptyset$.*

Proof. If $I = \emptyset$, this is clear. So, let $I \neq \emptyset$. If $X_{\mathbb{M}}(R)$ is connected, then so is R by [TG, I.11.2 Proposition 4], since the canonical morphism $X_{\mathbb{M}}(R) \to \mathrm{Spec}(R)$ is faithfully flat by 2.2.1 and in particular surjective. The converse follows by 2.4.2 and 2.4.1. □

(2.4.4) It can be seen from [5, Chapter 10] that 2.4.2 (and hence also 2.4.3) is true under less restrictive hypotheses on M than torsionfreeness and cancellability. Moreover, since the converse of 2.4.1 is obviously not true, the hypothesis on $\mathbb{M}$ under which 2.4.3 holds can probably be weakened somewhat more.

[8] By abuse of language we call a ring *connected* if its spectrum is connected.

2.5. Normality

To tackle normality in the affine case we use on one hand a result from [5] that forces us to restrict the monoids under consideration further, and on the other hand what we have done in 1.3.15 on rings of fractions of algebras of monoids.

(2.5.1) Proposition *Let M be a torsionfree, cancellable, finitely generated monoid. Then, $R[M]$ is integral and integrally closed if and only if R is integral and integrally closed.*

Proof. Since M is integrally closed by 1.2.4 b), this follows from 2.3.3 b) and [5, Corollary 12.11]. □

(2.5.2) Proposition *Let M be a torsionfree, cancellable, finitely generated monoid. Then, $R[M]$ is normal if and only if R is normal.*[9]

Proof. If $R[M]$ is normal, then 2.2.1 and [ÉGA, IV.2.1.13] imply that R is normal, too. Conversely, suppose that R is normal, and let $\mathfrak{p} \in \mathrm{Spec}(R[M])$. Then, $R_{\mathfrak{p}\cap R}$ is integral and integrally closed, and 2.5.1 implies that $R_{\mathfrak{p}\cap R}[M]$ is integral and integrally closed. By 1.3.15, $R[M]_{\mathfrak{p}}$ is a ring of fractions of $R_{\mathfrak{p}\cap R}[M]$ and hence integral and integrally closed by [AC, V.1.5 Proposition 16, Corollaire 1]. Therefore, $R[M]$ is normal. □

(2.5.3) Corollary *Let I be a lower semilattice, let $\mathbb{M} = (M_i)_{i\in I}$ be a projective system in* Mon *over I that is openly immersive for R, and suppose that M_i is torsionfree, cancellable and finitely generated for every $i \in I$. Then, $X_{\mathbb{M}}(R)$ is normal if and only if R is normal, or $I = \emptyset$.*

Proof. Clear from 2.5.2. □

(2.5.4) It can be seen from [5, Chapter 12] that the results in this section are true under less restrictive hypotheses on the monoids involved than the ones supposed here.

2.6. Chain conditions and dimension theory

Our first result on Noetherianity shows that monomial ideals behave in some way independent of the base ring.

(2.6.1) Proposition *Let M be a monoid. Then, the following statements are equivalent:*

(i) $R[M]$ is $\mathbb{T}_{R,M}$-Noetherian;

(ii) M is Noetherian, or $R = 0$.

[9] Concerning normality we use the terminology from [ÉGA], that is, a ring A is called *normal* if $A_{\mathfrak{p}}$ is integral and integrally closed for every $\mathfrak{p} \in \mathrm{Spec}(A)$.

Proof. The ring $R[M]$ is $\mathbb{T}_{R,M}$-Noetherian if and only if the ordered set $\mathbb{I}_{R,M}$ is Noetherian, and the monoid M is Noetherian if and only if the ordered set $\mathbb{I}_M$ is Noetherian. Hence, the claim is clear from 1.3.11. □

(2.6.2) Example Let E be a set. By 1.2.8 we know that $\mathbb{N}_0^{\oplus E}$ is Noetherian if and only if E is finite. Therefore, 2.6.1 implies that the R-algebra $R[\mathbb{N}_0^{\oplus E}]$ is $\mathbb{T}_{R,E}$-Noetherian if and only if E is finite, or if $R = 0$.

Next, we ask if Noetherianity and Artinianity are respected and reflected by the functor $X_{\mathbb{M}}(R)$. The case of Noetherianity is treated immediately, while the question concerning Artinianity will be answered in 2.6.9 on use of dimension theory.

(2.6.3) Proposition *Let M be a monoid.*

a) $R[M]$ is Noetherian if and only if R is Noetherian and M is finitely generated, or if $R = 0$.

b) $R[M]$ is Artinian if and only if R is Artinian and M is finite, or if $R = 0$.

Proof. a) holds by [5, Theorem 7.7], and b) holds by [5, Theorem 20.6]. □

(2.6.4) Corollary *Let I be a lower semilattice, and let $\mathbb{M} = (M_i)_{i\in I}$ be a projective system in* Mon *over I that is openly immersive for R. Consider the following statements:*

(1) $X_{\mathbb{M}}(R)$ is Noetherian;

(2) $X_{\mathbb{M}}(R)$ is locally Noetherian;

(3) R is Noetherian and M_i is finitely generated for every $i \in I$, or $R = 0$, or $I = \emptyset$.

a) It holds (1)⇒(2)⇔(3).

b) If I is finite, then it holds (1)⇔(2)⇔(3).

Proof. a) is clear by 2.6.3 a), and b) follows on use of 2.1.2 c). □

Now we turn to dimension theory. First we state that dimension theory of algebras of cancellable monoids comes down essentially to dimension theory of polynomial algebras, and then we derive formulas for the dimension of algebras of monoids and of schemes of the form $X_{\mathbb{M}}(R)$. Since (Krull-) dimension is not well adapted to non-Noetherian rings it is clear that assuming our base ring to be Noetherian yields nicer results.

(2.6.5) Proposition *Let M be a cancellable monoid. Then, it holds*

$$\dim(R[M]) = \dim\big(R[\mathbb{N}_0^{\oplus \operatorname{rk}(\operatorname{Diff}(M))}]\big).$$

Proof. By [5, Theorem 21.4; Theorem 17.1] we have $\dim(R[M]) = \dim(R[\operatorname{Diff}(M)]) = \dim(R[\mathbb{N}_0^{\oplus \operatorname{rk}(\operatorname{Diff}(M))}])$. □

(2.6.6) Proposition *Let M be a cancellable, finitely generated monoid, and let $n := \mathrm{rk}(\mathrm{Diff}(M))$.*

a) It holds

$$\dim(R) + n \leq \dim(R[M]) \leq \sum_{i=0}^{n-1} 2^i + 2^n \dim(R).$$

b) If R is Noetherian, then it holds

$$\dim(R) + n = \dim(R[M]).$$

Proof. a) By 2.6.5 we can assume without loss of generality that $M = \mathbb{N}_0^n$. The claim is trivial if $n = 0$ and holds by [AC, VIII.2.2 Proposition 3, Corollaire 2] if $n = 1$. So, let $n > 1$, and suppose the claim to be true for strictly smaller values of n. As $R[\mathbb{N}_0^n] \cong R[\mathbb{N}_0^{n-1}][\mathbb{N}_0]$ by 1.3.8, we get

$$\dim(R[\mathbb{N}_0^{n-1}]) + 1 \leq \dim(R[\mathbb{N}_0^n]) \leq 1 + 2\dim(R[\mathbb{N}_0^{n-1}])$$

and hence

$$\dim(R) + n = \dim(R) + (n-1) + 1 \leq \dim(R[\mathbb{N}_0^{n-1}]) + 1 \leq \dim(R[\mathbb{N}_0^n])$$

$$\leq 1 + 2\dim(R[\mathbb{N}_0^{n-1}]) \leq 1 + 2\Big(\sum_{i=0}^{n-2} 2^i + 2^{n-1}\dim(R)\Big) = \sum_{i=0}^{n-1} 2^i + 2^n \dim(R).$$

Thus, the claim follows by induction on n.

b) By 2.6.5 we can assume without loss of generality that $M = \mathbb{N}_0^n$, and then the claim holds by [AC, VIII.3.4 Proposition 7, Corollaire 3]. □

(2.6.7) Corollary *Let I be a lower semilattice, let $\mathbb{M} = (M_i)_{i\in I}$ be a projective system in* Mon *over I that is openly immersive for R, and suppose $I \neq \emptyset$, that M_i is cancellable and finitely generated for every $i \in I$, and that $n := \sup\{\mathrm{rk}(\mathrm{Diff}(M_i)) \mid i \in I\} \in \mathbb{N}_0$.*

a) It holds

$$\dim(R) + n \leq \dim(X_{\mathbb{M}}(R)) \leq \sum_{i=0}^{n-1} 2^i + 2^n \dim(R).$$

b) If R is Noetherian, then it holds

$$\dim(R) + n = \dim(X_{\mathbb{M}}(R)).$$

Proof. Clear from 2.6.6 and [ÉGA, IV.5.1.4]. □

(2.6.8) The examples given in [AC, VIII.2 Exercise 7] show that 2.6.6 b) and 2.6.7 b) need not hold if R is not Noetherian.

Using that Artinian rings are the same as Noetherian rings of dimension less or equal 0, we are now able to answer the question about the behaviour of Artinianity under the functor $X_{\mathbb{M}}(R)$.

(2.6.9) Corollary *Let I be a finite lower semilattice, let $\mathbb{M} = (M_i)_{i \in I}$ be a projective system in* Mon *over I that is openly immersive for R, and suppose that M_i is cancellable for every $i \in I$. Then, $X_{\mathbb{M}}(R)$ is Artinian if and only if R is Artinian and M_i is finite for every $i \in I$, or $R = 0$, or $I = \emptyset$.*

Proof. From 2.6.3 b) it is clear that Artinianity of $X_{\mathbb{M}}(R)$ implies the condition given above. Conversely, suppose that this condition holds. If $I = \emptyset$ or $R = 0$, then $X_{\mathbb{M}}(R)$ is obviously Artinian. So, let $I \neq \emptyset$ and $R \neq 0$. Since R is Artinian, it is also Noetherian by [ÉGA, I.2.8.2]. Moreover, for every $i \in I$ it holds $\mathrm{Diff}(M_i) = 0$. Thus, 2.6.7 b) yields $0 = \dim(R) = \dim(X_{\mathbb{M}}(R))$, and hence $X_{\mathbb{M}}(R)$ is Artinian by [ÉGA, I.2.8.2]. □

Finally, we give a condition under which equidimensionality is respected and reflected by the functor $X_{\mathbb{M}}$.

(2.6.10) Proposition *Let R be Noetherian, let I be a lower semilattice, let $\mathbb{M} = (M_i)_{i \in I}$ be a projective system in* Mon *over I that is openly immersive for R, and suppose that M_i is torsionfree, cancellable and finitely generated for every $i \in I$, and that $n := \sup\{\mathrm{rk}(\mathrm{Diff}(M_i)) \mid i \in I\} \in \mathbb{N}_0$. Then, $X_{\mathbb{M}}(R)$ is equidimensional if and only if R is equidimensional, or $I = \emptyset$.*

Proof. By [AC, II.4.3 Proposition 14, Corollaire 2], the ring R is equidimensional if and only if $\dim(R) = \dim(R/\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Min}(R)$. As R is Noetherian, it follows from 2.6.7 b) that this is the case if and only if

$$\dim(X_{\mathbb{M}}(R/\mathfrak{p})) = \dim(R/\mathfrak{p}) + n = \dim(R) + n$$

for every $\mathfrak{p} \in \mathrm{Min}(R)$. But then 2.3.12 implies that this is equivalent to $X_{\mathbb{M}}(R)$ being equidimensional. □

(2.6.11) Corollary *Let R be Noetherian, and let M be a torsionfree, cancellable, finitely generated monoid. Then, $R[M]$ is equidimensional if and only if R is equidimensional.*

Proof. Clear from 2.6.10. □

CHAPTER II

Cones and fans

In this chapter the combinatorial foundations for the theory of toric schemes are laid, that is, the theory of polycones and fans.

In Section 1 we start by collecting some basic facts and notations about structures on real vector spaces of finite dimension that are used in the following. We treat extensively *rational structures* with respect to a subring R of the field $\mathbb{R}$ of real numbers, providing a notion that includes the "lattice" used to define toric schemes. Then we define convex and conic sets, and we investigate some fundamental properties of them. The name *conic set* rather than *cone* is chosen since the latter is used in too many different senses throughout the literature. Similarly, we introduce the term *polycone*[1] for an intersection of finitely many closed linear halfspaces of a given real vector spaces of finite dimension, often called *polyhedral cone.* The main result on polycones is Theorem 1.4.5, saying that polycones and finitely generated conic sets are the same. At the end of the first section we introduce a notion of direct sum of polycones. Although this may seem naive, it turns out to give rise to a good theory of decomposition of polycones. More precisely, we show that every sharp polycone has a unique decomposition into indecomposable polycones. Besides this it will prove its usefulness some more times later on.

In Section 2, semifans and fans are defined. After studying projections of semifans (sometimes also known as quotient fans), we investigate certain topological properties of semifans. The main result in this direction is Theorem 2.3.17, giving a purely combinatorial description of the topological frontier of the support of a semifan – which is a good example of an easy-to-believe, but not-so-easy-to-prove statement as they are very common in the theory of polycones and fans. Finally, we treat subdivisions of fans. Using the notion of direct sum of polycones introduced above we prove that every fan has a simplicial subdivision.

Building on these foundations, Section 3 treats completions of semifans. As mentioned in the Introduction, we show that every (simplicial) semifan has a (simplicial) completion. This may be considered obvious sometimes (see, for example, the article [11]), but we will see that this is not at all the case. The proof given here is based on a sketch by Ewald and Ishida ([13]), itself based on a sketch by Ewald ([12]). I consider this section as a central part of the thesis, although it is used rarely in the later developement of toric

[1] analogous to *polytope*

schemes. Existence of completions is for sure interesting on its own, and the general approach pursued here certainly allows more specific questions, like existence of completions with some additional property **P** under the condition that the given semifan has property **P**.[2]

Finally, in Section 4 we introduce the combinatorial basis for our generalisation of Cox's theory of homogeneous coordinate ring. This relies on one hand on Cox's article [10], and on the other hand on the notion of Picard group of a fan as introduced by Ewald in [12].

1. Polycones

1.1. Structures on real vector spaces of finite dimension

Let us begin with the field of real numbers and the structures thereon that will be of importance in what follows.

(1.1.1) We denote by $\mathbb{R}$ the field of real numbers. We furnish $\mathbb{R}$ with its canonical ordering $\leq$ so that $\mathbb{R}$ is an ordered field. This ordering induces the canonical absolute value $|\cdot|$ so that $\mathbb{R}$ is a valued field. This absolute value induces the canonical topology so that $\mathbb{R}$ is a nondiscrete, separated, complete topological field.

Let $R \subseteq \mathbb{R}$ be a subring of $\mathbb{R}$. We furnish R with the ordering, absolute value and topology induced by $\mathbb{R}$ so that R is an ordered ring, a valued ring[3], and a separated topological ring. Let K denote the field of fractions of R, considered as a subfield of $\mathbb{R}$. Then, K is an ordered field, a valued field, and a nondiscrete, separated topological field.

The field $\mathbb{Q}$ of rational numbers is the prime subfield of $\mathbb{R}$, and the ordering, absolute value and topology induced by $\mathbb{R}$ are the canonical ones. Moreover, $\mathbb{R}$ is the completion of $\mathbb{Q}$ (as a topological ring), and hence $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore, $\mathbb{R}$ also is the completion of K (as a topological ring), K is dense in $\mathbb{R}$, and K is complete if and only if $K = \mathbb{R}$ ([TG, IV.1–3; IX.3.2], [A, VI.2]).

(1.1.2) An ordered group G is called *Archimedean* if for every $x \in G$ it holds $x \geq 0$ if and only if $\{nx \mid n \in \mathbb{N}\}$ is bounded below. If G is totally ordered, then it is Archimedean if and only if for all $x, y \in G_{>0}$ there exists

[2] One should note that in case of fans that are rational with respect to some $\mathbb{Z}$-structure there is another proof of the existence of completions, based on Sumihiro's Equivariant Compactification Theorem (see [20, 1.4 (pp 17–18)]): If Σ is a fan, then we can consider the toric variety $X_\Sigma(\mathbb{C})$ associated with Σ over $\mathbb{C}$, and the aforementioned theorem yields an embedding of $X_\Sigma(\mathbb{C})$ into a proper toric variety $X_{\Sigma'}(\mathbb{C})$ that is induced by an inclusion of the defining fans $\Sigma \subseteq \Sigma'$. Since properness of toric varieties is characterised by completeness of the defining fans (as we will see in IV.1.3), this argument also yields a completion of Σ. But it is quite unsatisfying to prove such an elementary statement about fans by a detour of this kind into algebraic geometry.

[3] generalising in an obvious way the notions of absolute value on a field and of valued field

an $r \in \mathbb{N}$ such that $rx > y$. An ordered ring is called *Archimedean* if its underlying additive ordered group is Archimedean. Since the ordered field[4] $\mathbb{R}$ is Archimedean, every subring of $\mathbb{R}$ has the same property ([A, VI.1 Exercises 31–33], [TG, IV.2.1]).

Next we look at affine spaces and vector spaces over $\mathbb{R}$. Although big parts of the theory to follow are rather affine than linear and hence could be developed in affine spaces, we mostly consider vector spaces to ease the exposition. Moreover, we restrict ourselves mostly to vector spaces of finite dimension in order to have a satisfying duality (1.1.5) and a canonical topology (1.1.6), but nothing will be lost since the object of our studies – polycones and fans – are of finite dimension by nature.

(1.1.3) Let V be an affine $\mathbb{R}$-space. We denote by $\dim_{\mathbb{R}}(V)$ or, if no confusion can arise, by $\dim(V)$ the dimension of V. Note that the dimension of the empty affine $\mathbb{R}$-space is $-\infty$.

(1.1.4) Let V be an $\mathbb{R}$-vector space. By abuse of language we denote the affine $\mathbb{R}$-space underlying V again by V. For $A \subseteq V$ we denote by $\langle A \rangle_{\mathbb{R}}$ or, if no confusion can arise, by $\langle A \rangle$ the sub-$\mathbb{R}$-vector space of V generated by A. Moreover, we denote the zero $\mathbb{R}$-vector space by 0.

For $A \subseteq V$ we write $-A := \{-x \mid x \in A\}$. For $A \subseteq V$ and $R \subseteq \mathbb{R}$ we write $RA := \{rx \mid r \in R \wedge x \in A\}$, and for $x \in V$ and $r \in \mathbb{R}$ we write $Rx := R\{x\}$ and $rA := \{r\}A$. For a finite family $(A_i)_{i \in I}$ of subsets of V we write

$$\sum_{i \in I} A_i := \{\sum_{i \in I} a_i \mid (a_i)_{i \in I} \in \prod_{i \in I} A_i\}.$$

If one of the sets A_i is of the form $\{x\}$, then we write in the above notation x instead of $\{x\}$. Note that the sum of the empty family of subsets of V is 0.

(1.1.5) Let V be an $\mathbb{R}$-vector space. We denote by V^* the dual space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of V and by V^{**} the bidual space $(V^*)^*$ of V. Moreover, we denote by $c_V : V \to V^{**}$ the canonical morphism in $\mathsf{Mod}(\mathbb{R})$. For $A \subseteq V$ we call the sets

$$A^{\perp,V} := \{u \in V^* \mid \forall x \in A : u(x) = 0\},$$

$$A^{\vee,V} := \{u \in V^* \mid \forall x \in A : u(x) \geq 0\}$$

and

$$A^{\circ,V} := \{u \in V^* \mid \forall x \in A : u(x) \geq -1\}$$

respectively *the orthogonal of A, the dual of A,* and *the polar of A.* If no confusion can arise, we denote these sets by $A^{\perp}$, $A^{\vee}$ and A°. For $x \in V$ we write $x^{\perp}$, $x^{\vee}$ and x° instead of $\{x\}^{\perp}$, $\{x\}^{\vee}$ and $\{x\}^{\circ}$.

[4]Keep in mind that ordered fields are totally ordered by definition.

Now, suppose that $\dim(V)$ is finite. Then $c_V : V \to V^{**}$ is an isomorphism, and by means of this we identify V and V^{**}. In particular, we consider the orthogonal, the dual and the polar of a subset of V^* as a subset of V. For $u \in V^*$ and $r \in \mathbb{R}$ we set

$$H_{=r}(u) := \{x \in V \mid u(x) = r\},$$

and we define analogously the subsets $H_{\leq r}(u)$, $H_{<r}(u)$, $H_{\geq r}(u)$ and $H_{>r}(u)$ of V. For $A \subseteq V^*$ it holds

$$A^{\perp} = \bigcap_{u \in A} H_{=0}(u), \quad A^{\vee} = \bigcap_{u \in A} H_{\geq 0}(u), \quad \text{and } A^{\circ} = \bigcap_{u \in A} H_{\geq -1}(u).$$

For $A \subseteq V$ it holds $A^{\perp} = \langle A \rangle^{\perp}$, and $A^{\perp}$ is a sub-$\mathbb{R}$-vector space of V^*. Moreover, the map $A \mapsto A^{\perp}$ induces an antiisomorphism of ordered sets from the set of sub-$\mathbb{R}$-vector spaces of V to the set of sub-$\mathbb{R}$-vector spaces of V^*, its inverse being given by $A \mapsto A^{\perp}$. For a sub-$\mathbb{R}$-vector space $A \subseteq V$ it holds

$$\dim(A) + \dim(A^{\perp}) = \dim(V),$$

and for a family $(A_i)_{i \in I}$ of sub-$\mathbb{R}$-vector spaces of V it holds

$$\Big(\sum_{i \in I} A_i\Big)^{\perp} = \bigcap_{i \in I} A_i^{\perp} \text{ and } \Big(\bigcap_{i \in I} A_i\Big)^{\perp} = \sum_{i \in I} A_i^{\perp}.$$

Finally, for a sub-$\mathbb{R}$-vector space $A \subseteq V$ it holds $A^{\perp} = A^{\vee} = A^{\circ}$ ([A, II.2.4; II.2.7; II.7.5]).

(1.1.6) Let V be an $\mathbb{R}$-vector space of finite dimension. We furnish V with its canonical topology, that is, the only topology on V turning it into a separated topological $\mathbb{R}$-vector space[5], and then V is complete. Moreover, if V' is a topological $\mathbb{R}$-vector space and $f : V \to V'$ is a morphism in $\mathsf{Mod}(\mathbb{R})$, then f is continuous. If $V' \subseteq V$ is a sub-$\mathbb{R}$-vector space, then it is closed, and the topologies on V' and V/V' induced by V are the canonical ones. Furthermore, V is normable, and all norms on V are equivalent and hence induce the canonical topology. More precisely, if $\|\cdot\|$ and $\|\cdot\|'$ are norms on V, then there is an $\alpha \in \mathbb{R}_{\geq 1}$ such that for every $x \in V$ it holds $\frac{1}{\alpha}||x|| \leq ||x||' \leq \alpha||x||$.

The canonical isomorphism $c_V : V \xrightarrow{\cong} V^{**}$ in $\mathsf{Mod}(\mathbb{R})$ is an isomorphism in $\mathsf{TopMod}(\mathbb{R})$. Moreover, the canonical bilinear form on $V \times V^*$ sets V and V^* in duality, and the weak topologies defined by this are the canonical ones ([EVT, I.1.1; I.2.3; II.6.1–2], [TG, IX.3.3]).

If $x \in V$, then a subset $U \subseteq V$ containing x is called *symmetric with respect to* x if for every $y \in V$ with $x + y \in U$ it holds $x - y \in U$. If $x \in V$, then it is clear that every neighbourhood of x in V contains a neighbourhood of x in V that is symmetric with respect to x.

[5] Equivalently, this is the topology on V obtained by transporting the product topology on $\mathbb{R}^{\dim(V)}$ to V by means of an (arbitrary) isomorphism of $\mathbb{R}$-vector spaces $\mathbb{R}^{\dim(V)} \cong V$.

As mentioned above we start now to introduce rational structures with respect to a subring R of $\mathbb{R}$. The applications in mind concern the case $R = \mathbb{Z}$, whereas the theory given in [A, II.8] supposes R to be a field. We give a slight generalisation of Bourbaki's approach, adapted especially to the finite dimensional case.

(1.1.7) Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, and let V be an $\mathbb{R}$-vector space of finite dimension. By means of scalar restriction we get a structure of R-module on V, and by abuse of language we denote the resulting R-module again by V. For a free R-module W we denote by $\mathrm{rk}_R(W)$ its rank. If $W \subseteq V$ is a free sub-R-module, then the canonical morphism $K \otimes_R W \to V$ in $\mathsf{Mod}(K)$ with $a \otimes x \mapsto ax$ is a monomorphism, and its image is a sub-K-vector space of V, denoted by W_K. An *R-structure on V* is a free sub-R-module $W \subseteq V$ such that $\mathrm{rk}_R(W) = \dim_{\mathbb{R}}(V)$ and that $\langle W \rangle_{\mathbb{R}} = V$. A free sub-$R$-module $W \subseteq V$ is an R-structure on V if and only if the canonical morphism $\mathbb{R} \otimes_R W \to V$ in $\mathsf{Mod}(\mathbb{R})$ with $a \otimes x \mapsto ax$ is an isomorphism, and this is the case if and only if W_K is a K-structure on V. In this case W_K is called *the K-structure on V induced by W.*

The R-module underlying R is an R-structure on the $\mathbb{R}$-vector space underlying $\mathbb{R}$, and if I is a finite set, then the product R-module R^I is canonically an R-structure on the product $\mathbb{R}$-vector space $\mathbb{R}^I$. Hence, if W is a K-structure on V, furnished with its canonical topology, then V is the completion of W and hence W is dense in V ([A, II.8.1]).

(1.1.8) Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, and let W be an R-structure on V. An element $x \in V$ is called *W-rational* if it lies in W. An element $x \in V$ is W_K-rational if and only if there is an $r \in R \setminus 0$ such that rx is W-rational.

A sub-$\mathbb{R}$-vector space $V' \subseteq V$ is called *W-rational* if it has a basis consisting of W-rational elements. This is the case if and only if V' is W_K-rational. Then, $W \cap V'$ is an R-structure on V', called *the induced R-structure,* and it holds

$$(W \cap V')_K = W_K \cap V'.$$

An affine sub-$\mathbb{R}$-space $V' \subseteq V$ is called *W-rational* if it is the translation of a W-rational sub-$\mathbb{R}$-vector space of V by a W-rational element of V. An affine sub-$\mathbb{R}$-space $V' \subseteq V$ is W_K-rational if and only if there is an $r \in R \setminus 0$ such that rV' is W-rational.

If $V' \subseteq V$ is a W-rational sub-$\mathbb{R}$-vector space and W' is the induced R-structure on V', then the R-module W/W' is canonically isomorphic to and identified with an R-structure on V/V', called *the induced R-structure,* and it holds $(W/W')_K = W_K/W'_K$. If no confusion can arise we denote the R-structure W/W' on V/V' by W/V' ([A, II.8.1–3]).

(1.1.9) Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V and V' be $\mathbb{R}$-vector spaces of finite dimension, and let W and W' be R-structures on V and V' respectively. A morphism $f : V \to V'$ in $\mathsf{Mod}(\mathbb{R})$ is called *rational with respect to* W *and* W' if it induces by restriction and coastriction a morphism of R-modules $W \to W'$. The morphism f is rational with respect to W_K and W'_K if and only if there is an $r \in R \setminus 0$ such that rf is rational with respect to W and W'. If f is rational with respect to W and W', then its kernel is W-rational and its image is W'-rational. If $V' \subseteq V$ is a W-rational sub-$\mathbb{R}$-vector space and W' the induced R-structure on V', then the canonical injection $V' \hookrightarrow V$ is rational with respect to W' and W, and the canonical projection $V \twoheadrightarrow V/V'$ is rational with respect to W and W/V'.

A morphism $f : V \to V'$ of affine $\mathbb{R}$-spaces is called *rational with respect to* W *and* W' if its associated morphism in $\mathsf{Mod}(\mathbb{R})$ is rational with respect to W and W' and if moreover $f(0)$ is W'-rational. The morphism f is rational with respect to W_K and W'_K if and only if there is an $r \in R \setminus 0$ such that rf is rational with respect to W and W' ([A, II.8.3]).

(1.1.10) Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, and let W be an R-structure on V. A linear form $u \in V^*$ is called W*-rational* if it is rational with respect to W and R. We denote by W^* the set of W-rational linear forms on V which is an R-structure on V^* and canonically isomorphic to the dual module $\mathrm{Hom}_R(W, R)$ of the R-module W. Applying this to V^* instead of V, we get the R-structure $W^{**} := (W^*)^*$ on V^{**}, and it is readily checked that the canonical isomorphism $c_V : V \to V^{**}$ in $\mathsf{Mod}(\mathbb{R})$ induces by restriction and coastriction the canonical isomorphism $c_W : W \to W^{**}$ in $\mathsf{Mod}(R)$ and hence is rational with respect to W and W^{**}. So, in identifying V and V^{**} we identify also their R-structures W and W^{**}.

A linear form $u \in V^*$ is W_K-rational if and only if there is an $r \in R \setminus 0$ such that ru is W-rational, and hence the K-structures $(W_K)^*$ and $(W^*)_K$ on V coincide ([A, II.8.4]).

(1.1.11) Let $R \subseteq \mathbb{R}$ be a subring and let K denote the field of fractions of R. If $(V_i)_{i \in I}$ is a finite family of $\mathbb{R}$-vector spaces of finite dimension and W_i is an R-structure on V_i for every $i \in I$, then $W := \bigoplus_{i \in I} W_i$ is canonically isomorphic to and identified with an R-structure on $V := \bigoplus_{i \in I} V_i$, and the canonical injection $V_i \hookrightarrow V$ in $\mathsf{Mod}(\mathbb{R})$ is rational with respect to W_i and W for every $i \in I$. Moreover, it holds

$$\Big(\bigoplus_{i \in I} W_i\Big)_K = \bigoplus_{i \in I} (W_i)_K.$$

Suppose now that V is an $\mathbb{R}$-vector space of finite dimension, that W is an R-structure on V and that V_i is a W-rational sub-$\mathbb{R}$-vector space of V and W_i is the induced R-structure on V_i for every $i \in I$. Then, the canonical morphism

$$\bigoplus_{i\in I} V_i \to V,\ (x_i)_{i\in I} \mapsto \sum_{i\in I} x_i$$

in $\mathsf{Mod}(\mathbb{R})$ is rational with respect to $\bigoplus_{i\in I} W_i$ and W.

If V' is an $\mathbb{R}$-vector space of finite dimension, W' is an R-structure on V', and $f : V \twoheadrightarrow V'$ is an epimorphism in $\mathsf{Mod}(\mathbb{R})$ that is rational with respect to W and W', then there exists a section of f in $\mathsf{Mod}(\mathbb{R})$ that is rational with respect to W' and W. Hence, if V' is a W-rational sub-$\mathbb{R}$-vector space of V, then there exists a W-rational complement of V' in V.

A fundamental notion in the theory of convexity is separation by hyperplanes. We collect here the necessary definitions and basic observations.

(1.1.12) Let $R \subseteq \mathbb{R}$ be a subring, let V be an $\mathbb{R}$-vector space of finite dimension, and let W be an R-structure on V. An *affine W-hyperplane in V* is a W-rational affine sub-$\mathbb{R}$-space of V of codimension 1, and a *linear W-hyperplane in V* is a W-rational sub-$\mathbb{R}$-vector space of V of codimension 1. Note that the codimension of the empty affine sub-$\mathbb{R}$-space of the zero space 0 is ∞ and hence it is not considered to be an affine hyperplane in 0.

A subset $H \subseteq V$ is a linear W_K-hyperplane in V if and only if it is a linear W-hyperplane in V, and it is an affine, nonlinear W_K-hyperplane in V if and only if there is an $r \in R \setminus 0$ such that rH is an affine, nonlinear W-hyperplane in V. A subset $H \subseteq V$ is a linear W-hyperplane in V if and only if it is of the form $H_{=0}(u) = u^\perp$ for some $u \in W^* \setminus 0$, and it is an affine, nonlinear W-hyperplane in V if and only if it is of the form $H_{=r}(u)$ for some $u \in W^* \setminus 0$ and $r \in R \setminus 0$, or – equivalently – of the form $u^\perp + x$ for some $u \in W^* \setminus 0$ and $x \in W \setminus u^\perp$ ([A, II.8.4]).

(1.1.13) Let $R \subseteq \mathbb{R}$ be a subring, let V be an $\mathbb{R}$-vector space of finite dimension, and let W be an R-structure on V.

Let H be an affine W-hyperplane in V. Then, $V \setminus H$ has two connected components U and U'. Each of these is open and is called an *open affine W-halfspace in V defined by H.* Their closures are $U \cup H$ and $U' \cup H$, and each of them is called a *closed affine W-halfspace in V defined by H.* Moreover, U and U' respectively are the interiors of $U \cup H$ and $U' \cup H$. If H is a linear W-hyperplane in V, then an open or closed affine W-halfspace in V defined by H respectively is called an *open (linear) W-halfspace in V defined by H* or a *closed (linear) W-halfspace in V defined by H.*

An *open affine W-halfspace in V* or a *closed affine W-halfspace in V* respectively is an open or closed affine W-halfspace in V defined by an affine W-hyperplane in V. An *open (linear) W-halfspace in V* or a *closed (linear) W-halfspace in V* respectively is an open or closed linear W-halfspace in V defined by a linear W-hyperplane in V.

Let $U \subseteq V$ be a subset. Then, U is a closed (or open) linear W_K-halfspace in V if and only if it is a closed (or open) linear W-halfspace in

V. Moreover, U is a closed (or open) affine, nonlinear W_K-halfspace in V if and only if there is an $r \in R \setminus 0$ such that rU is a closed (or open) affine, nonlinear W-halfspace in V.

Let $U \subseteq V$ be a subset. Then, U is a closed linear W-halfspace in V if and only if it is of the form $H_{\geq 0}(u) = u^{\vee}$ for some $u \in W^* \setminus 0$, and it is an open linear W-halfspace in V if and only if it is of the form $H_{>0}(u) = u^{\vee} \setminus u^{\perp}$ for some $u \in W^* \setminus 0$. Moreover, U is a closed (or open) affine, nonlinear W-halfspace in V containing 0 if and only if it is of the form $H_{\geq r}(u)$ (or $H_{>r}(u)$) for some $u \in W^* \setminus 0$ and $r \in R_{<0}$, or – equivalently – of the form $u^{\vee}+x$ (or $u^{\vee} \setminus u^{\perp}+x$) for some $u \in W^* \setminus 0$ and $x \in W \cap H_{<0}(u)$. Furthermore, U is a closed (or open) affine, nonlinear W-halfspace in V not containing 0 if and only if it is of the form $H_{\geq r}(u)$ (or $H_{>r}(u)$) for some $u \in W^* \setminus 0$ and $r \in R_{>0}$, or – equivalently – of the form $u^{\vee} + x$ (or $u^{\vee} \setminus u^{\perp} + x$) for some $u \in W^* \setminus 0$ and $x \in W \cap H_{>0}(u)$ ([TG, VI.1.4]).

(1.1.14) Let $R \subseteq \mathbb{R}$ be a subring, let V be an $\mathbb{R}$-vector space of finite dimension, and let W be an R-structure on V. Let H be an affine hyperplane in V. A subset $A \subseteq V$ is said to *lie (strictly) on one side of* H if it is contained in one of the closed (or open, respectively) affine halfspaces of V defined by H. Two subsets $A, B \subseteq V$ are said to *lie (strictly) on different sides of* H or to be *(strictly) separated by* H if they are contained in different closed (or open, respectively) affine halfspace of V defined by H, and they are said to be *separated in their intersection by* H if they are separated by H and if moreover

$$A \cap H = A \cap B = B \cap H.$$

Two subsets $A, B \subseteq V$ are called *W-separable, strictly W-separable,* or *W-separable in their intersection* respectively, if there exists an affine W-hyperplane H in V such that A and B are separated, strictly separated, or separated in their intersection by H.

If $A \subseteq V$ and if H is an affine hyperplane in V such that A lies on one side of H, then $\mathrm{in}(A)$ lies strictly on one side of H.

Real vector spaces of finite dimension can be furnished with more structure than we have considered up to now, for example with a Hilbert norm. We end this section with some remarks about Hilbert norms. Since purely algebraic or topological arguments are usually easier and better suited for revealing what happens, we try to avoid the use of norms. Still they will appear in the proofs (but not in the statements) of 1.2.20, 3.6.8 and 3.6.9.

(1.1.15) Let V be an $\mathbb{R}$-vector space of finite dimension, and let W be an R-structure on V. As V is complete, Hilbert norms on V correspond to separating, positive, symmetric $\mathbb{R}$-bilinear forms on V. A Hilbert norm on V is called *W-rational* if its corresponding separating, positive, symmetric $\mathbb{R}$-bilinear form on V induces by restriction and coastriction an R-bilinear form on W. There exists a W-rational Hilbert norm on V.

Now, let $\|\cdot\|$ be a W-rational Hilbert norm on V, and let f be the corresponding separating, positive, symmetric $\mathbb{R}$-bilinear form on V. Let $U \subseteq V$ be a W-rational sub-$\mathbb{R}$-vector space. Then, the norm on U induced by $\|\cdot\|$ is a $W \cap U$-rational Hilbert norm on U, and its corresponding separating, positive, symmetric $\mathbb{R}$-bilinear form on U is induced by f.

Let $V' := V/U$, let $W' := W/U$, and let $p : V \twoheadrightarrow V'$ denote the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$. Let $\overline{U}$ denote the orthogonal complement of U in V with respect to $\|\cdot\|$, that is, the sub-$\mathbb{R}$-vector space of V consisting of those $x \in V$ with the property that $f(x, y) = 0$ for all $y \in U$. Since

$$\overline{U} = \bigcap_{y \in U} \mathrm{Ker}(f(\cdot, y)) = \bigcap_{y \in W \cap U} \mathrm{Ker}(f(\cdot, y)),$$

this sub-$\mathbb{R}$-vector space of V is W-rational. Moreover, it holds $V = U \oplus \overline{U}$, and denoting by $q : V \twoheadrightarrow \overline{U}$ the canonical projection in $\mathsf{Mod}(\mathbb{R})$, which is rational with respect to W and $W \cap \overline{U}$, there exists a unique isomorphism $h : V' \xrightarrow{\cong} \overline{U}$ in $\mathsf{Mod}(\mathbb{R})$ such that $h \circ p = q$. In particular, h is rational with respect to W' and $W \cap \overline{U}$. The W-rational Hilbert norm $\|\cdot\|$ on V induces a $W \cap \overline{U}$-rational Hilbert norm $\overline{\|\cdot\|}$ on $\overline{U}$, and since U is closed in V it induces also a norm $\|\cdot\|'$ on V'. Then, $h : V' \to \overline{U}$ is an isomorphism of normed $\mathbb{R}$-vector spaces with respect to $\|\cdot\|'$ and $\overline{\|\cdot\|}$, and hence $\|\cdot\|'$ is a W'-rational Hilbert norm on V'.

1.2. Convex and conic sets

Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, and let W be an R-structure on V.

On use of the structure of ordered field on $\mathbb{R}$ we may impose conditions on linear combinations. In this way we are lead to the notions of convex and conic combinations, and thus by taking closures with respect to these kind of combinations to convex and conic sets.

(1.2.1) Let $A \subseteq V$. A *convex combination of* A is an $\mathbb{R}$-linear combination $\sum_{x \in A} r_x x$ of A such that $r_x \geq 0$ for every $x \in A$ and that $\sum_{x \in A} r_x = 1$. A *conic combination of* A is an $\mathbb{R}$-linear combination $\sum_{x \in A} r_x x$ of A such that $r_x \geq 0$ for every $x \in A$. The sets of convex or conic combinations of A respectively are denoted by $\mathrm{conv}(A)$ and $\mathrm{cone}(A)$, and they are called *the convex hull of* A and *the conic hull of* A. We call A *convex* or *conic* respectively if it is closed under formation of convex or conic combinations, that is, if $A = \mathrm{conv}(A)$ or $A = \mathrm{cone}(A)$. Since intersections of convex or conic sets respectively are convex or conic it is clear that $\mathrm{conv}(A)$ and $\mathrm{cone}(A)$ are the smallest convex or conic subsets of V containing A. Moreover, conic sets are convex.

For $x, y \in V$ we denote by $[\![x, y]\!]$ the convex hull of $\{x, y\}$, and we set $]\!]x, y]\!] := [\![x, y]\!] \setminus \{x\}$ and $]\!]x, y[\![:= [\![x, y]\!] \setminus \{x, y\}$. A subset $A \subseteq V$ is convex if and only if $[\![x, y]\!] \subseteq A$ for all $x, y \in A$, and it is conic if and only if it is nonempty and fulfils $A + A \subseteq A$ and $\mathbb{R}_{\geq 0}A \subseteq A$ ([EVT, II.2.1 Proposition 1; II.2.4 Proposition 10; Proposition 11, Corollaire][6]).

(1.2.2) The topological $\mathbb{R}$-vector space V is locally convex, that is, every point of V has a fundamental system of neighbourhoods consisting of convex sets ([EVT, II.4.2]).

(1.2.3) The maps $A \mapsto \mathrm{conv}(A)$ and $A \mapsto \mathrm{cone}(A)$ are closures[7] on the ordered set of subsets of V. Moreover, for $A \subseteq V$ we have

$$\mathrm{cone}(\mathrm{conv}(A)) = \mathrm{conv}(\mathrm{cone}(A)) = \mathrm{cone}(A)$$

and $\mathrm{conv}(A) \subseteq \mathrm{cone}(A)$. If $A \subseteq V$ is nonempty, then it follows easily from [EVT, II.2.4 Proposition 12] that $\mathrm{cone}(A) = \mathbb{R}_{\geq 0} \mathrm{conv}(A)$.

(1.2.4) Let $A \subseteq V$. We call $A \subseteq V$ *(finitely) W-conic* if there is a (finite) $B \subseteq W$ such that $A = \mathrm{cone}(B)$, and if no confusion can arise every such set B is called a *W-generating set of A.* In case $W = V$ we speak just of (finitely) conic sets and of generating sets of such. A generating set B of A is called *minimal* if no proper subset of B is a generating set of A. Obviously, every finitely W-conic subset of V has a minimal W-generating set.

Clearly, A is (finitely) W_K-conic if and only if it is (finitely) W-conic. Moreover, if A is W-conic, then the sub-$\mathbb{R}$-vector space $\langle A \rangle$ of V is W-rational.

(1.2.5) Let V' be a further $\mathbb{R}$-vector space of finite dimension, let W' be an R-structure on V', and let $f : V \to V'$ be a morphism in $\mathsf{Mod}(\mathbb{R})$ that is rational with respect to W and W'. For $A \subseteq V$ it holds $f(\mathrm{conv}(A)) = \mathrm{conv}(f(A))$, and images and preimages under f of convex sets are again convex. Furthermore, for $A \subseteq V$ it holds $f(\mathrm{cone}(A)) = \mathrm{cone}(f(A))$, and images and preimages under f of conic sets are again conic. Moreover, images under f of (finitely) W-conic sets are (finitely) W'-conic.

For $x \in V$ and $A \subseteq V$ it holds $\mathrm{conv}(A) + x = \mathrm{conv}(A + x)$, and hence if A is convex, then so is $A + x$ ([EVT, II.2.1 Proposition 2; II.2.4]).

(1.2.6) Example Affine sub-$\mathbb{R}$-spaces of V and closed or open affine halfspaces in V are convex, and W-rational sub-$\mathbb{R}$-vector spaces of V and closed linear W-halfspaces in V are finitely W-conic ([EVT, II.2.6 Proposition 16, Corollaire 1]).

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[6] In the terminology of [EVT], A is conic if and only if it is a pointed convex cone with summit in 0.

[7] If E is an ordered set, then a *closure on* E is an idempotent endomorphism f of E with $\mathrm{Id}_E \leq f$, where E^E is furnished with the product ordering.

(1.2.7) Let $(A_i)_{i\in I}$ be a finite family of subsets of V. Then, it holds

$$\mathrm{cone}(\bigcup_{i\in I} A_i) = \sum_{i\in I} \mathrm{cone}(A_i).$$

Hence, if A_i is (finitely) W-conic for every $i \in I$, then so is $\sum_{i\in I} A_i$.

(1.2.8) For $A \subseteq V$ it holds $A^\vee = \mathrm{cone}(A)^\vee$, and $A^\vee$ is a closed conic subset of V^*. Moreover, the map $A \mapsto A^\vee$ induces an antiisomorphism of ordered sets from the set of closed conic subsets of V to the set of closed conic subsets of V^*, its inverse being given by $A \mapsto A^\vee$. For a finite family $(A_i)_{i\in I}$ of conic subsets of V it holds

$$\Big(\sum_{i\in I} A_i\Big)^\vee = \bigcap_{i\in I} A_i^\vee,$$

and for a conic subset $A \subseteq V$ it holds $A^\vee = A^\circ$ ([EVT, II.6.3 Proposition 4; Théorème 1]).

(1.2.9) Since the values of the map $A \mapsto A^\vee$ are closed sets, the antiisomorphism from 1.2.8 cannot be extended to the set of not necessarily closed, conic subsets of V.

To describe very roughly the size and the shape of a conic set A we can consider the smallest vector space containing A and the greatest vector space contained in A, respectively. These ideas are useful and will be considered next, leading in particular to the notion of sharpness.

(1.2.10) If $A \subseteq V$ is convex, then the dimension of the affine sub-$\mathbb{R}$-space of V generated by A is denoted by $\dim(A)$ and called *the dimension of A,* and A is called *full (in V)* if $\dim(A) = n$.

Now, suppose that $A \subseteq V$ is conic. Then, it holds $\langle A\rangle = A - A$ by [EVT, II.2.4 Proposition 10, Corollaire 1]. Moreover, $s(A) := A \cap (-A)$ is the greatest sub-$\mathbb{R}$-vector space of V contained in A by [EVT, II.2.4 Proposotion 10, Corollaire 2], called *the summit of* A, and A is called *sharp* if $s(A) = 0$. If A is W-conic and B is a W-generating set of A, then it holds $s(A) = \langle x \in B \mid -x \in A\rangle$, and hence $s(A)$ is W-rational. If $f : V \twoheadrightarrow V/s(A)$ denotes the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$, then $f(A)$ is a sharp conic subset of $V/s(A)$.

If A is moreover closed, then it is easily seen on use of 1.1.5 and 1.2.8 that A is sharp if and only if $A^\vee$ is full, and vice versa.

The next result gives a sharpness criterion for conic sets in terms of a generating set.

(1.2.11) Proposition *Let $A \subseteq V$. Then, the following statements are equivalent:*

(i) $0 \notin A$, *and* $\mathrm{cone}(A)$ *is sharp;*

(ii) $0 \notin \mathrm{conv}(A)$.

Proof. We set $B := \operatorname{cone}(A)$ and $C := \operatorname{conv}(A)$. First, suppose that $0 \notin A$ and that B is sharp, and assume that $0 \in C$. Then, there is a family $(r_x)_{x\in A}$ of finite support in $\mathbb{R}_{\geq 0}$ with $\sum_{x\in A} r_x = 1$ such that $\sum_{x\in A} r_x x = 0$, and hence an $x_0 \in A$ with $r_{x_0} \neq 0$. From this we get the contradiction $x_0 = -\sum_{x\in A\setminus\{x_0\}} \frac{r_x}{r_{x_0}} x \in B \cap (-B) = s(B) = 0$.

Conversely, suppose that $0 \notin C$ and hence $0 \notin A$, and assume that B is not sharp. Then, there is an $x \in A$ with $-x \in B$ by 1.2.10, and it follows from 1.2.3 that there is an $r \in \mathbb{R}_{>0}$ with $-rx \in C$. But this yields the contradiction $0 \in [\![x, -rx]\!] \subseteq C$. □

The following theorem is an application of the geometric form of the Theorem of Hahn-Banach. It will be used later at important points (1.2.15, 1.4.4), and so the Theorem of Hahn-Banach can be considered fundamental for the whole theory developed here. We have to begin with a topological remark.

(1.2.12) Let X be a topological space, and let $A, B \subseteq X$. If B is connected and meets A and $X \setminus A$, then it also meets $\operatorname{fr}(A)$ ([TG, I.11.1 Proposition 3]).

Convex subsets of V are connected by [EVT, II.2.6 Remarque]. Therefore, if $A \subseteq V$, $x \in A$ and $y \in V \setminus A$, then the above implies that $[\![x, y]\!]$ meets $\operatorname{fr}(A)$.

(1.2.13) Theorem *Let A and B be disjoint, nonempty subsets of V.*

a) Suppose that A is convex and open, and that B is an affine sub-$\mathbb{R}$-space of V. Then, there exists an affine hyperplane $H \subseteq V$ containing B such that A lies strictly on one side of H.

b) Suppose that A is convex and closed, and that B is convex and compact. Then, A and B are strictly separable.

Proof. a) By the geometric form of the Theorem of Hahn-Banach ([EVT, II.5.1 Théorème 1]) there exists an affine hyperplane $H \subseteq V$ containing B and not meeting A, and the claim follows on use of 1.2.12.

b) holds by 1.2.2 and [EVT, II.5.3 Proposition 4]. □

We saw in 1.2.8 and 1.2.9 that closedness of conic sets is necessary for the map $A \mapsto A^\vee$ to induce a reasonable duality. The next task is to show that finitely conic sets are closed. We do this similar to [EVT, II.7.3 Proposition 6].

(1.2.14) Lemma *Let X be a topological space, let $(A_i)_{i\in I}$ be a family of closed subsets of X, and suppose that for every $x \in X$ there exist a neighbourhood U of x and a $j \in I$ such that $U \cap \bigcup_{i\in I} A_i \subseteq A_j$. Then, $\bigcup_{i\in I} A_i$ is closed.*

Proof. Let $A := \bigcup_{i\in I} A_i$, let $x \in \operatorname{cl}(A)$, and let U be a neighbourhood of x and $j \in I$ such that $U \cap A \subseteq A_j$. If V is a neighbourhood of x, then

$U \cap V$ is a neighbourhood of x, hence meets A, and therefore $U \cap V$ meets A_j. This implies $x \in \mathrm{cl}(A_j)$, and as A_j is closed it follows $x \in A_j \subseteq A$. □

(1.2.15) Proposition *a) If $(A_i)_{i \in I}$ is a finite family of compact, convex subsets of V, then $\mathrm{conv}(\bigcup_{i \in I} A_i)$ is compact.*

b) If $A \subseteq V$ is compact and convex with $0 \notin A$, then $\mathrm{cone}(A)$ is closed.

Proof. a) holds by [EVT, II.2.6 Proposition 15]. To show b), we set $B := \mathrm{cone}(A)$. By 1.2.13 b), there exists an affine hyperplane $H \subseteq V$ that separates 0 and A strictly. Let $u \in V^* \setminus 0$ and $r \in \mathbb{R}_{>0}$ such that $H = H_{=r}(u)$. We set $L := [0,1]A$, and it is readily checked that $L = \mathrm{conv}(A \cup \{0\})$. Hence, L is compact by a). Moreover, it holds $B = \mathbb{R}_{\geq 1}L$ by 1.2.3, and it is easily seen that $H \cap L = H \cap B$. Therefore, $H \cap B$ is compact. Furthermore it holds $B = \mathbb{R}_{\geq 0}(H \cap B)$.

For every $k \in \mathbb{N}$ we set $B_k := [0,k](H \cap B)$, and then we have $B_k = \mathrm{conv}(k(H \cap B) \cup \{0\})$. This set is compact by a). Since $B = \bigcup_{k \in \mathbb{N}} B_k$, it follows from 1.2.14 that is suffices to show that for every $x \in V$ there exists a neighbourhood U of x and a $k \in \mathbb{N}$ with $U \cap B \subseteq B_k$. So, let $x \in V$. As $\mathbb{R}$ is Archimedean, there is a $k \in \mathbb{N}$ with $kr > u(x)$. Setting $U := H_{<kr}(u) = kH$ we see that U is a neighbourhood of x, and moreover it holds $U \cap B = (kH) \cap B = k(H \cap B) \subseteq B_k$ as desired. □

(1.2.16) Corollary *If $A \subseteq V$ is finite, then $\mathrm{conv}(A)$ is compact and $\mathrm{cone}(A)$ is closed.*

Proof. The first statement is clear by 1.2.15 a). To prove the second statement, we set $B := \mathrm{cone}(A)$ and denote by $f : V \twoheadrightarrow V/s(B)$ the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$. Then, $f(B) \subseteq V/s(B)$ is sharp and finitely conic by 1.2.5 and 1.2.10. As $B = f^{-1}(f(B))$, it suffices to show that $f(B)$ is closed, and thus we may assume that B is sharp. Since $B = \mathrm{cone}(A \setminus \{0\})$ we may moreover assume that $0 \notin A$. Hence, it follows from 1.2.11 that $0 \notin \mathrm{conv}(A)$. As $\mathrm{conv}(A)$ is compact by the first statement and $B = \mathrm{cone}(\mathrm{conv}(A))$, the claim follows from 1.2.15 b). □

(1.2.17) Corollary *Let $A \subseteq V$ be sharp finitely conic, let $H \subseteq V$ be a linear hyperplane such that A lies on one side of H and that $H \cap A = 0$, and let $p \in A \setminus \{0\}$. Then, $(H + p) \cap A$ is compact.*

Proof. This was shown in the proof of 1.2.15. □

(1.2.18) Let X be a topological space. A subset $A \subseteq X$ is called *nowhere dense (in X)* if $\mathrm{in}(\mathrm{cl}(A)) = \emptyset$. If $A \subseteq X$ is nowhere dense in X and $B \subseteq A$, then B is nowhere dense in X, and if $(A_i)_{i \in I}$ is a finite family of subsets of V that are nowhere dense in X, then $\bigcup_{i \in I} A_i$ is nowhere dense in X. Finally, if $A \subseteq X$ is closed, then it is nowhere dense in X if and only if $\mathrm{fr}(A) = A$ ([TG, IX.5.1]).

(1.2.19) Example Every proper affine sub-$\mathbb{R}$-space of V is nowhere dense in V ([TG, IX.5.1 Exemple 3]).

The next proposition characterises topologically those convex sets that are full. We will be have plenty of occasions to use this result in the following, and we start here by giving a series of important applications.

(1.2.20) Proposition *A convex subset $A \subseteq V$ is full if and only if* $\mathrm{in}(A)$ *is nonempty.*

Proof. If A is not full, then it is contained in a proper affine sub-$\mathbb{R}$-space of V and hence nowhere dense in V by 1.2.18 and 1.2.19, and thus it holds $\mathrm{in}(A) = \emptyset$.

Conversely, suppose that A is full. Without loss of generality we can suppose that $0 \in A$. There exists a basis E of V contained in A, and we set $a := \frac{1}{n+1}\sum_{e\in E} e$; as this is a convex combination of $E \cup \{0\}$ it lies in A. Let $\|\cdot\|$ denote the 1-norm on V defined by E, and let B denote the open ball with center a and radius $\frac{1}{2(n+1)}$ with respect to $\|\cdot\|$. It suffices to show that $B \subseteq A$. So, let $x \in B$. Then, there is a family $(r_e)_{e\in E}$ in $\mathbb{R}$ with $x = \sum_{e\in E} r_e e$. For every $e \in E$ it holds

$$|r_e - \tfrac{1}{n+1}| \leq ||x - a|| < \tfrac{1}{2(n+1)} < \tfrac{1}{n+1}$$

and hence $r_e \in \mathbb{R}_{\geq 0}$. Moreover, it holds

$$\textstyle\sum_{e\in E} r_e = ||x|| \leq ||a|| + ||x - a|| < \frac{1}{n+1} + \frac{1}{2(n+1)} = \frac{2n+1}{2n+2} < 1$$

and thus $x \in \mathrm{conv}(E \cup \{0\}) \subseteq A$ as desired. □

(1.2.21) Let $\mathbb{A}$ be a finite set of closed convex subsets of V. Then, $\bigcup \mathbb{A}$ is closed, and it is nowhere dense in V if and only if no $A \in \mathbb{A}$ is full, as is seen on use of 1.2.18 and 1.2.20.

(1.2.22) Lemma *Let X be a topological space, and let $A \subseteq X$. Then, it holds*

$$\mathrm{fr}(\mathrm{cl}(A)) \cup \mathrm{fr}(\mathrm{in}(A)) \subseteq \mathrm{fr}(A).$$

Proof. Let $x \in \mathrm{fr}(\mathrm{cl}(A)) \cup \mathrm{fr}(\mathrm{in}(A))$, and let U be a neighbourhood of x. If $x \in \mathrm{fr}(\mathrm{cl}(A))$, then U meets $\mathrm{cl}(A)$ and $\mathrm{in}(X \setminus A)$, hence A and $X \setminus A$. If $x \in \mathrm{fr}(\mathrm{in}(A))$, then U meets $\mathrm{in}(A)$ and $\mathrm{cl}(X \setminus A)$, hence A and $X \setminus A$. This proves the claim. □

(1.2.23) Proposition *a) If $A \subseteq V$ is convex, then so are* $\mathrm{cl}(A)$ *and* $\mathrm{in}(A)$.

b) If $A \subseteq V$ is convex and full, then it holds $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{in}(A))$, $\mathrm{in}(A) = \mathrm{in}(\mathrm{cl}(A))$, *and* $\mathrm{fr}(A) = \mathrm{fr}(\mathrm{in}(A))$.

c) If $A \subseteq V$ is conic, then so are $\mathrm{cl}(A)$ *and* $\mathrm{in}(A) \cup \{0\}$.

Proof. This follows easily from 1.2.20, [EVT, II.2.6 Proposition 16, Corollaires 1–2; Proposition 14] and 1.2.22. □

(1.2.24) Corollary *Let $A, B \subseteq V$, and suppose that A is convex, that B is open, and that A meets B. Then,* $\mathrm{in}_{\langle A\rangle}(A)$ *meets B.*

Proof. We assume that $\mathrm{in}_{\langle A\rangle}(A)$ does not meet B. Then, there is an $x \in \mathrm{fr}_{\langle A\rangle}(A) \cap B$, and hence there is a neighbourhood U of x in V contained in B. But it holds $x \in \mathrm{fr}_{\langle A\rangle}(A) = \mathrm{fr}_{\langle A\rangle}(\mathrm{in}_{\langle A\rangle}(A))$ by 1.2.23 b), and as $U \cap \langle A\rangle$ is a neighbourhood of x in $\langle A\rangle$ we get the contradiction that $\mathrm{in}_{\langle A\rangle}(A) \cap U$ is a nonempty subset of $\mathrm{in}_{\langle A\rangle}(A) \cap B$. □

(1.2.25) Corollary *Let $A \subseteq V$ be convex, closed and full, and let $x \in$* $\mathrm{fr}(A)$*. Then, there exists an affine hyperplane $H \subseteq V$ containing x such that A lies on one side of H.*

Proof. We know from 1.2.20 and 1.2.23 a) that $\mathrm{in}(A)$ is nonempty, open, and convex, and as $x \notin \mathrm{in}(A)$ it follows from 1.2.13 a) that there is an affine hyperplane $H \subseteq V$ containing x such that $\mathrm{in}(A)$ lies strictly on one side of H. On use of 1.2.23 b) we see that $A = \mathrm{cl}(\mathrm{in}(A))$ lies on one side of H, and this implies the claim. □

(1.2.26) If $A \subseteq V$ is convex and H is an affine hyperplane in V, then A lies strictly on one side of H if and only if A does not meet H ([EVT, II.2.2 Proposition 4]). If $A \subseteq V$ is conic, and $u \in V^*$ and $r \in R$ are such that $A \subseteq H_{>r}(u)$, then it holds $A \subseteq H_{\geq 0}(u)$ ([EVT, II.5.3 Lemme 1]).

(1.2.27) Proposition *Let $A \subseteq V$, and let $x \in A$. We consider the following statements:*

(1) $x \in \mathrm{in}_{\langle A\rangle}(A)$;

(2) $A^\vee \cap x^\perp = A^\perp$;

(3) $\langle A\rangle = A - \mathrm{cone}(x)$;

(4) $A \subseteq \mathrm{cone}(x) - A$.

a) It holds (1)⇒(2)⇐(3)⇒(4).

b) If A is conic, then it holds (1)⇔(2)⇐(3)⇔(4).

Proof. First, let $x \in \mathrm{in}_{\langle A\rangle}(A)$, let $u \in A^\vee \cap x^\perp$, and assume that $u \notin A^\perp$. Setting $\overline{u} := u \upharpoonright_{\langle A\rangle}$ we see that $\overline{u}^{\perp,\langle A\rangle}$ is a linear hyperplane in $\langle A\rangle$ containing x such that A lies on one side of it, contradictory to 1.1.14. The other inclusion being obvious, this shows that (1) implies (2).

Second, if $\langle A\rangle = A - \mathrm{cone}(x)$, then it follows on use of 1.2.8 that

$$A^\perp = \langle A\rangle^\perp = (A - \mathrm{cone}(x))^\vee = A^\vee \cap (-\mathrm{cone}(x))^\vee = A^\vee \cap x^\perp,$$

and hence (3) implies (2). Moreover, it obviously implies (4).

Now, let A be conic. Suppose that $A^\vee \cap x^\perp = A^\perp$, and assume that $x \notin \mathrm{in}_{\langle A\rangle}(A)$. Then, it holds $x \in \mathrm{fr}_{\langle A\rangle}(\mathrm{cl}_{\langle A\rangle}(A))$, and as $\mathrm{cl}_{\langle A\rangle}(A)$ is closed and conic by 1.2.23 c), it follows from 1.2.25 and 1.2.26 that there is a $\overline{u} \in \langle A\rangle^* \setminus 0$ such that $\overline{u}(x) = 0$ and that $\mathrm{cl}_{\langle A\rangle}(A) \subseteq \overline{u}^{\vee,\langle A\rangle}$. There is a linear form u on V coinciding with $\overline{u}$ on $\langle A\rangle$, and it is readily checked that $u \in A^\vee \cap x^\perp \setminus A^\perp$, contradictory to our hypothesis. Thus, (2) implies (1).

Finally, suppose that $A \subseteq \mathrm{cone}(x) - A$. Then, 1.2.10 yields

$$\langle A\rangle = A - A \subseteq A - \mathrm{cone}(x) \subseteq \langle A\rangle,$$

and hence (4) implies (3). □

We end this section with two constructions involving convexity that will be used in 2.3.2 and 2.2.7, respectively.

(1.2.28) Let L be an affine line in V, let $x, y, z \in L$ be such that $y \in \rrbracket x, z\rrbracket$, and let U be a neighbourhood of z. Then, it holds $y \in \mathrm{in}(\mathrm{conv}(\{x\} \cup U))$. Indeed, by translation with $-x$ we can assume that $x = 0$, and then there is an $r \in]0, 1]$ with $y = rz$. Therefore rU is a neighbourhood of y, and it is readily checked that $rU \subseteq \mathrm{conv}(\{0\} \cup U)$.

(1.2.29) Let $\mathbb{A}$ be a finite set of closed, convex subsets of V, and let $x, y \in V$ with $x \neq y$ be such that $\llbracket x, y\rrbracket \subseteq \bigcup \mathbb{A}$. Then, there exist an $A \in \mathbb{A}$ and a $z \in \rrbracket x, y\rrbracket$ such that $\llbracket x, z\rrbracket \subseteq A$. Indeed, since x is not contained in the closed set $\bigcup\{A \in \mathbb{A} \mid x \notin A\}$, there is a neighbourhood U of x in V such that every element of $\mathbb{A}$ that is met by U contains x. Moreover, there is a $z \in \rrbracket x, y\rrbracket \cap U$, and therefore there is an $A \in \mathbb{A}$ containing z and hence meeting U. Then, convexity of A implies $\llbracket x, z\rrbracket \subseteq A$ as claimed.

1.3. Faces of conic sets

Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, and let W be an R-structure on V.

With the definition of faces of conic sets, combinatorics will enter the field. After some general observations we show that finitely conic sets have only finitely many faces.

(1.3.1) Let $A \subseteq V$ be conic. A *face of* A is a subset $B \subseteq A$ such that there exists a $u \in A^\vee$ with $B = A \cap u^\perp$. By taking $u = 0$ it is seen that A is a face of A. Since sub-$\mathbb{R}$-vector spaces of V are conic it is clear that faces of A are conic. A face B of A is called *proper* if $B \neq A$. We denote by $\mathrm{face}(A)$ and $\mathrm{pface}(A)$ respectively the sets of faces and of proper faces of A. If B and C are faces of A, then it is readily checked that $B \cap C$ is a face of A again, and it is moreover a face of B and of C. For $k \in \mathbb{N}_0$ we set

$$\mathrm{face}(A)_k := \{B \in \mathrm{face}(A) \mid \dim(B) = k\},$$

and if no confusion can arise we denote this set also by A_k.

We denote by $C \preccurlyeq B$ the relation "C is a face of B" on the set of conic subsets of V. If $A \subseteq V$ is conic, then the above shows that the relations $C \preccurlyeq B$ and $C \subseteq B$ on $\mathrm{face}(A)$ coincide.

(1.3.2) Proposition *Let $B \subseteq A \subseteq V$, and let $u \in V^*$ be such that $B \subseteq u^\perp$ and that $A \setminus B \subseteq u^\vee \setminus u^\perp$. Then, it holds* $\mathrm{cone}(A) \subseteq u^\vee$ *and* $\mathrm{cone}(B) = \mathrm{cone}(A) \cap u^\perp$.

Proof. The first claim and the inclusion $\mathrm{cone}(B) \subseteq \mathrm{cone}(A) \cap u^\perp$ are clear. So, let $w \in \mathrm{cone}(A) \cap u^\perp$. Then, there is a family $(r_x)_{x \in A}$ of finite support in $\mathbb{R}_{\geq 0}$ such that $w = \sum_{x \in A} r_x x$. It holds

$$\sum_{x \in A \setminus B} r_x u(x) = \sum_{x \in A} r_x u(x) = u(w) = 0,$$

and as $u(x) > 0$ for every $x \in A \setminus B$ this implies $r_x = 0$ for every $x \in A \setminus B$. Thus, w is a conic combination of B, and hence the claim is proven. □

(1.3.3) Corollary *Let $A \subseteq V$ be W-conic, let X be a W-generating set of A, and let $B \in$* $\mathrm{face}(A)$. *Then, B is W-conic, and there exists a subset of X that is a W-generating set of B.*

Proof. If $u \in A^\vee$ fulfils $B = A \cap u^\perp$, then it follows from 1.3.2 that $B = \mathrm{cone}(X \cap u^\perp)$. □

(1.3.4) Corollary *Let $A \subseteq V$ be finitely W-conic. Then, every face of A is finitely W-conic, and* $\mathrm{face}(A)$ *is finite.*

Proof. Clear by 1.3.3. □

The following results prepare the ground for projections of fans, treated in 2.2.

(1.3.5) Let $A \subseteq V$ be conic. Then, it holds $s(A) \subseteq \bigcap \mathrm{face}(A)$, and if $0 \in \mathrm{face}(A)$, then A is sharp. Indeed, for $x \in s(A)$ and $u \in A^\vee$ it holds $x \in A \cap u^\perp$, and this implies the claim.

(1.3.6) Proposition *Let V' be an $\mathbb{R}$-vector space of finite dimension, let $f : V \to V'$ be a morphism in* $\mathsf{Mod}(\mathbb{R})$, *let $A \subseteq V$ be conic, and let $B \in$* $\mathrm{face}(A)$.

a) If $\mathrm{Ker}(f) \subseteq \langle B \rangle$, *then it holds* $f(B) \in \mathrm{face}(f(A))$.

b) If $\mathrm{Ker}(f) = \langle B \rangle$, *then $f(A)$ is sharp.*

Proof. a) There is a $u \in A^\vee$ such that $B = A \cap u^\perp$. We set $\overline{V} := V/\mathrm{Ker}(f)$ and denote by $p : V \twoheadrightarrow \overline{V}$ the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$. It holds $\mathrm{Ker}(f) \subseteq u^\perp$, and hence there is a $u' \in \overline{V}^*$ with $u' \circ p = u$. Moreover, there is a monomorphism $f' : \overline{V} \rightarrowtail V'$ in $\mathsf{Mod}(\mathbb{R})$ such that $f' \circ p = f$. Let $s : V' \to \overline{V}$ be a retraction of f' in $\mathsf{Mod}(\mathbb{R})$, and set $v := u' \circ s \in (V')^*$. Then, we have $v \circ f = u$, and it is readily checked that $f(A) \subseteq v^\vee$ and that $f(B) = f(A) \cap v^\perp$. Thus, $f(B)$ is a face of $f(A)$.

b) From a) we get that $0 \in \mathrm{face}(A)$, and hence $f(A)$ is sharp by 1.3.5. □

(1.3.7) Proposition *Let $A \subseteq V$ be conic, let $V' := V/\langle A \rangle$, and let f denote the canonical epimorphism $V \twoheadrightarrow V'$ in* $\mathsf{Mod}(\mathbb{R})$.

a) If $B \subseteq V$ is conic with $A \preccurlyeq B$, then it holds

$$\mathrm{face}(f(B)) = \{f(C) \mid A \preccurlyeq C \preccurlyeq B\}.$$

b) If $B, B' \subseteq V$ are conic with $A \preccurlyeq B$ and $A \preccurlyeq B'$, then it holds

$$f(B) \cap f(B') = f(B \cap B').$$

c) If $B \subseteq V$ is conic with $A \preccurlyeq B$, then the map

$$\{C \mid A \preccurlyeq C \preccurlyeq B\} \to \mathrm{face}(f(B)),\ C \mapsto f(C)$$

is an isomorphism of ordered sets.

Proof. a) First, suppose that $A \preccurlyeq C \preccurlyeq B$. Then, there is a $u \in B^\vee$ with $C = B \cap u^\perp$, and clearly we have $\mathrm{Ker}(f) = \langle A \rangle \subseteq u^\perp$. Therefore, there exists a $v \in (V')^*$ such that $v \circ f = u$. It is readily checked that $v \in f(B)^\vee$ and that $f(C) = f(B) \cap v^\perp$, and thus it holds $f(C) \preccurlyeq f(B)$.

Now, suppose that $\overline{C} \preccurlyeq f(B)$. Then, there is a $v \in f(B)^\vee$ with $\overline{C} = f(B) \cap v^\perp$. We set $u := v \circ f \in V^*$ and $C := B \cap u^\perp$. Clearly we have $u \in B^\vee$ and hence $C \preccurlyeq B$. It is readily checked that $\overline{C} = f(C)$ and that $A \subseteq B \cap u^\perp = C$ and hence $A \preccurlyeq C$. From this we get the claim.

b) Let $x \in (B - A) \cap (B' - A)$. Then, there are $y \in B$, $y' \in B'$ and $z, z' \in A$ with $y - z = x = y' - z'$. This implies $x = (y + z') - (z' + z)$, and as $y + z' = y' + z \in B \cap B'$ we get $x \in (B \cap B') - A$. Hence, it holds $(B - A) \cap (B' - A) = (B \cap B') - A$, and from this it follows

$$f(B) \cap f(B') = f(f^{-1}(f(B) \cap f(B'))) = f((B - A) \cap (B' - A)) =$$

$$f((B \cap B') - A) = f(B \cap B').$$

c) From a) we know that this map exists and is surjective. Moreover, it is obviously increasing. Suppose that $A \preccurlyeq C \preccurlyeq B$ and $A \preccurlyeq C' \preccurlyeq B$, and that $f(C) = f(C')$. By b) we have $f(C) = f(C \cap C')$ and hence

$$\dim(C) - \dim(A) = \dim(f(C)) =$$

$$\dim(f(C \cap C')) = \dim(C \cap C') - \dim(A).$$

This yields $\dim(C) = \dim(C \cap C')$ and hence $C = C \cap C' \preccurlyeq C'$. Now, by reasons of symmetry it follows $C = C'$, and therefore the map in question is injective, hence bijective.

Finally, suppose that $\overline{C} \preccurlyeq \overline{C'} \preccurlyeq f(B)$. By a) there are $C, C' \in \mathrm{face}(B)$ with $A \preccurlyeq C$ and $A \preccurlyeq C'$ such that $\overline{C} = f(C)$ and $\overline{C'} = f(C')$, and on use of b) it follows

$$f(C) = \overline{C} = \overline{C} \cap \overline{C}' = f(C) \cap f(C') = f(C \cap C').$$

Now, injectivity implies $C = C \cap C'$ and hence $C \preccurlyeq C'$, and thus we see that the inverse of the above map is increasing, too. Therefore, the claim is proven. $\square$

Next we give a combinatorial description of the topological frontier of a closed conic set. This gives a first idea about how the frontier of a semifan might look like – a question solved completely in 2.3.17.

(1.3.8) Proposition *If $A \subseteq V$ is closed and conic, then it holds*

$$\mathrm{fr}_{\langle A\rangle}(A) = \bigcup \mathrm{pface}(A).$$

Proof. By 1.2.27 b) we have

$$\mathrm{fr}_{\langle A\rangle}(A) = \{x \in A \mid A^\vee \cap x^\perp \neq A^\perp\} = \bigcup_{u \in A^\vee \setminus A^\perp} A \cap u^\perp = \bigcup \mathrm{pface}(A). \quad \square$$

(1.3.9) Corollary *Let $A \subseteq V$ be closed and conic, let $(u_B)_{B\in\mathrm{face}(A)}$ be a family in $A^\vee$, and suppose that for every $B \in \mathrm{face}(A)$ it holds $B = A \cap u_B^\perp$. Then, it holds*

$$A = \Big(\bigcap_{B\in\mathrm{face}(A)} u_B^\vee\Big) \cap \langle A\rangle.$$

Proof. Obviously, we have $A \subseteq (\bigcap_{B\in\mathrm{face}(A)} u_B^\vee) \cap \langle A\rangle$. Let $x \in \langle A\rangle \setminus A$. As A is full in $\langle A\rangle$, there exists a $z \in \mathrm{in}_{\langle A\rangle}(A)$ by 1.2.20, and 1.2.12 implies that $]\!]x, z[\![$ meets $\mathrm{fr}_{\langle A\rangle}(A)$. Therefore, by 1.3.8 there are a $C \in \mathrm{pface}(A)$ and a $t \in]0,1[$ such that $y := tx + (1-t)z \in C$. The choice of u_C implies

$$tu_C(x) + (1-t)u_C(z) = u_C(y) = 0.$$

But t, $1-t$ and $u_C(z)$ being strictly positive we get $u_C(x) < 0$ and hence $x \notin \bigcap_{B\in\mathrm{face}(A)} u_B^\vee$. Herewith the claim is proven. $\square$

(1.3.10) Corollary *Let $A, B \subseteq V$, and suppose that A is closed and conic, that $A \cap B \preccurlyeq A$, and that $\mathrm{in}_{\langle A\rangle}(A)$ meets B. Then, it holds $A \subseteq B$.*

Proof. If $A \cap B$ is a proper face of A, then we get the contradiction $\emptyset \neq \mathrm{in}_{\langle A\rangle}(A) \cap B \subseteq A \cap B \subseteq \bigcup \mathrm{pface}(A) = \mathrm{fr}_{\langle A\rangle}(A)$ on use of 1.3.8. So, it follows $A = A \cap B \subseteq B$. $\square$

(1.3.11) Corollary *Let $\mathbb{A}$ be a set of closed, conic subsets of V and let $\mathbb{B}$ be a set of subsets of V such that for every $A \in \mathbb{A}$ and every $B \in \mathbb{B}$ it holds $A \cap B \preccurlyeq A \not\subseteq B$. Then, it holds $(\bigcup \mathbb{A}) \cap \mathrm{in}(\bigcup \mathbb{B}) = \emptyset$.*

Proof. We assume that $\bigcup \mathbb{A}$ meets $\mathrm{in}(\bigcup \mathbb{B})$. Then, there is an $A \in \mathbb{A}$ meeting $\mathrm{in}(\bigcup \mathbb{B})$, and 1.2.24 implies that $\mathrm{in}_{\langle A\rangle}(A)$ meets $\mathrm{in}(\bigcup \mathbb{B})$ and hence $\bigcup \mathbb{B}$. Thus, there is a $B \in \mathbb{B}$ meeting $\mathrm{in}_{\langle A\rangle}(A)$. But then 1.3.10 implies the contradiction $A \subseteq B$. $\square$

The next result is also known by the name of *Orthogonality Theorem.* It states that taking duals is compatible with the facial structure on closed conic sets.

(1.3.12) Proposition *Let $A \subseteq V$ be closed and conic. Then, there are mutually inverse antiisomorphisms of ordered sets*

$$\mathrm{face}(A) \to \mathrm{face}(A^\vee),\ B \mapsto A^\vee \cap B^\perp$$

and

$$\mathrm{face}(A^\vee) \to \mathrm{face}(A),\ B \mapsto A \cap B^\perp.$$

Proof. Let $B \in \mathrm{face}(A)$. By 1.2.20 there exists a $y \in \mathrm{in}_{\langle B\rangle}(B)$, and by 1.2.8 it holds $y \in A = A^{\vee\vee}$. Moreover, 1.2.27 b) implies that $A^\vee \cap B^\perp = A^\vee \cap B^\vee \cap y^\perp = A^\vee \cap y^\perp$, and hence it holds $A^\vee \cap B^\perp \in \mathrm{face}(A^\vee)$. Thus, on use of 1.2.8 we see that both maps exist. Moreover, by 1.2.8 it is clear that they are decreasing. To show that they are mutually inverse let $B \in \mathrm{face}(A)$, and let $u \in A^\vee$ be such that $B = A \cap u^\perp$ and hence $u \in A^\vee \cap B^\perp$. By 1.2.8 we have $B \subseteq A \cap B^{\perp\perp} \subseteq A \cap (A^\vee \cap B^\perp)^\perp \subseteq A \cap u^\perp = B$, and this yields the claim. □

(1.3.13) Proposition *Let $A, B \subseteq V$ be conic with $B \preccurlyeq A$. Then, it holds $A \cap \langle B\rangle = B$.*

Proof. There is a $u \in A^\vee$ such that $B = A \cap u^\perp$, and it follows $\langle B\rangle \subseteq u^\perp$, hence $A \cap \langle B\rangle \subseteq A \cap u^\perp = B \subseteq A \cap \langle B\rangle$, and thus the claim. □

(1.3.14) Proposition *a) Let A, B, C and D be conic subsets of V such that* $\dim(A) = 1$*, that $A \not\subseteq D$ and that $C \cap D \preccurlyeq C \supseteq A \cup B$. Then, it holds*

$$D \cap (A + B) = D \cap B.$$

b) Let A, B, B', C and C' be conic subsets of V such that $\dim(A) = 1$*, that $A \not\subseteq B'$, that $A \not\subseteq B$, that $C \cap B' \preccurlyeq C \supseteq A \cup B$ and that $C' \cap B \preccurlyeq C' \supseteq A \cup B'$. Then, it holds*

$$(A + B) \cap (A + B') = A + (B \cap B').$$

Proof. a) As $A \cup B \subseteq C$ we have $A + B \subseteq C$ and therefore $D \cap (A+B) = (C \cap D) \cap (A + B)$. As $C \cap D \preccurlyeq C$, we can replace D by $C \cap D$ and thus assume without loss of generality that $D \preccurlyeq C$. Hence, there is a $u \in C^\vee$ with $D = C \cap u^\perp$. Let $x \in D \cap (A + B)$. Then, there are $y \in A$ and $z \in B$ with $x = y + z$, and it follows $0 = u(x) = u(y) + u(z)$. But $y, z \in C \subseteq u^\vee$ implies $u(y) = u(z) = 0$ and therefore $y, z \in D$. In particular, it holds $y \in A \cap D$. Now, we assume that $A \cap D \neq 0$. Then, $A \cap D$ being conic and contained in A the hypotheses imply that A is not sharp. Hence, C is not sharp, too, yielding together with 1.3.5 the contradiction $A \subseteq s(C) \subseteq D$. Therefore, it holds $y = 0$ and hence $x = z \in D \cap B$. The other inclusion being obvious, the claim is proven.

b) Let $x \in (A + B) \cap (A + B')$. Then, there are $y, y' \in A$, $z \in B$ and $z' \in B'$ such that $y + z = x = y' + z'$ and hence $y - y' \in \langle A\rangle = A \cup (-A)$. Therefore, it holds $y - y' \in A$ or $y' - y \in A$, hence $z = z' + y' - y \in B \cap (A + B')$ or $z' = z + y - y' \in B' \cap (A + B)$, and thus $z \in B \cap B'$ or $z' \in B \cap B'$ by a). This implies $x \in A + (B \cap B')$. The other inclusion being obvious, the claim is proven. $\square$

(1.3.15) Lemma *Let A, B and C be conic subsets of V such that $C \subseteq B$ and that $A \cap B \preccurlyeq B$.*

a) It holds $A \cap C \preccurlyeq C$.

b) If $\dim(A) = \dim(A \cap C)$, then it holds $A \preccurlyeq B$.

Proof. There exists $u \in B^\vee \subseteq C^\vee$ with $A \cap B = B \cap u^\perp$, and hence it holds $C \cap u^\perp = C \cap B \cap u^\perp = C \cap A \cap B = A \cap C$, thus $A \cap C \preccurlyeq C$. If moreover $\dim(A) = \dim(A \cap C)$, then we get $A = A \cap C \subseteq C \subseteq B$ and hence $A = A \cap B \preccurlyeq B$. $\square$

1.4. Polycones

Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, and let W be an R-structure on V.

Now the main characters of the first part of this chapter appear, polycones (also known as polyhedral cones), defined as intersections of finite families of closed linear halfspaces.

(1.4.1) A *W-polycone (in V)* is the intersection of a finite family of closed linear W-halfspaces in V. Clearly, W-polycones are closed and conic, and intersections of W-polycones are again W-polycones. If $A \subseteq V$ is a $W \cap \langle A \rangle$-polycone, then it is easily seen by choosing a W-rational complement of $\langle A \rangle$ in V (1.1.11) that A is a W-polycone. Furthermore, $A \subseteq V$ is a W_K-polycone in V if and only if it is a W-polycone in V. In case $W = V$ we speak just of polycones in V.

(1.4.2) Example Every W-halfspace in V and every W-rational sub-$\mathbb{R}$-vector space of V is a W-polycone in V.

The first aim is to prove what might be called the *Fundamental Theorem on Polycones,* that is, that W-polycones are the same as finitely W-generated conic sets. After some preparations we will achieve this in 1.4.5.

(1.4.3) Proposition *Let $A \subseteq V$ be finitely W-conic, and let $B \in \operatorname{face}(A)$. Then, there exists a $u \in W^* \cap A^\vee$ such that $B = A \cap u^\perp$.*

Proof. Without loss of generality we can assume that $R = K$. Let X be a finite W-generating set of A. By 1.3.3 there is a subset $Y \subseteq X$ that is a W-generating set of B. Moreover, there exists $u \in A^\vee$ such that $B = A \cap u^\perp$.

For $Z \subseteq X$ we consider the morphism

$$f_Z : V^* \to \mathbb{R}^Z,\ v \mapsto (v(x))_{x \in Z}$$

in $\mathsf{Mod}(\mathbb{R})$. From 1.1.7 we know that R^Z is an R-structure on $\mathbb{R}^Z$, and it is clear that f_Z is rational with respect to W^* and R^Z. Therefore, $\operatorname{Ker}(f_Z)$

is W^*-rational by 1.1.9. Clearly, it holds $u \in \mathrm{Ker}(f_Y)$. Furthermore, X being finite implies that $(\mathbb{R}_{>0})^{X\setminus Y}$ is an open neighbourhood of $f_{X\setminus Y}(u)$ in $R^{X\setminus Y}$, and $f_{X\setminus Y}$ being continuous hence yields that $U := f_{X\setminus Y}^{-1}((\mathbb{R}_{>0})^{X\setminus Y})$ is an open neighbourhood of u in V^*. Thus, $U \cap \mathrm{Ker}(f_Y)$ is a nonempty, open subset of $\mathrm{Ker}(f_Y)$. As $W^* \cap \mathrm{Ker}(f_Y)$ is an R-structure on $\mathrm{Ker}(f_Y)$ and therefore dense in $\mathrm{Ker}(f_Y)$ by 1.1.7, there exists a $v \in W^* \cap \mathrm{Ker}(f_Y) \cap U$.

Now, we show that $A \subseteq v^\vee$ and that $B = A \cap v^\perp$. As $(v(x))_{x\in Y} = f_Y(v) = 0$ and hence $v(x) = 0$ for every $x \in Y$, we get $Y \subseteq v^\perp$. Furthermore, since $(v(x))_{x\in X\setminus Y} = f_{X\setminus Y}(v) \in (\mathbb{R}_{>0})^{X\setminus Y}$ and hence $v(x) > 0$ for every $x \in X \setminus Y$, we get $X \setminus Y \subseteq v^\vee \setminus v^\perp$. Thus, our claim follows from 1.3.2. □

(1.4.4) Proposition *Let $(u_i)_{i\in I}$ be a finite family in V^*, and let $A := \bigcap_{i\in I} u_i^\vee$. Then, it holds $A^\vee = \mathrm{cone}(\{u_i \mid i \in I\})$.*

Proof. We set $B := \mathrm{cone}(\{u_i \mid i \in I\})$. Clearly, we have $u_i \in A^\vee$ for every $i \in I$, and as $A^\vee$ is conic by 1.2.8 we get $B \subseteq A^\vee$. Conversely, let $u \in V^* \setminus B$. By 1.2.16, 1.2.13 b) and 1.2.26 there exists a linear hyperplane $H \subseteq V^*$ such that B lies on one side of H and that u lies strictly on the other side of H, that is, an $x \in V \setminus 0$ such that $B \subseteq H_{\geq 0}(x)$ and that $u \in H_{<0}(x)$. It follows $u_i(x) \geq 0$ for every $i \in I$ and hence $x \in A$. But as $u(x) < 0$ this implies that $u \notin A^\vee$, and thus we have $A^\vee = B$. □

(1.4.5) Theorem *A subset $A \subseteq V$ is finitely W-conic if and only if it is a W-polycone.*

Proof. If A is finitely W-conic, then it follows from 1.3.4, 1.4.3, 1.2.16 and 1.3.9 that A is a W-polycone. Conversely, let A be a W-polycone. Then, $A^\vee$ is finitely W^*-conic by 1.4.4 and hence a W^*-polycone by the above. Again applying 1.4.4 we see on use of 1.2.8 that $A = A^{\vee\vee}$ is finitely W-conic, and thus the claim is proven. □

(1.4.6) On use of 1.4.5 we may translate previous results on finitely conic subsets into results on polycones and vice versa. In particular, we have the following: the intersection of finitely many finitely W-conic subsets of V is finitely W-conic (1.4.1); the sum of finitely many W-polycones in V is a W-polycone (1.2.7); if A is a W-polycone in V, then $A^\vee$ is a W^*-polycone in V^* (1.4.4).

On use of the above and 1.2.8 we see that if $(A_i)_{i\in I}$ is a finite family of polycones in V, then it holds $(\bigcap_{i\in I} A_i)^\vee = \sum_{i\in I} A_i^\vee$.

(1.4.7) Corollary *Let A be a polycone in V, and let $x \in A$. Then, the following statements are equivalent:*

(i) $x \in \mathrm{in}_{\langle A\rangle}(A)$;

(ii) $A^\vee \cap x^\perp = A^\perp$;

(iii) $\langle A\rangle = A - \mathrm{cone}(x)$;

(iv) $A \subseteq \mathrm{cone}(x) - A$.

Proof. By 1.2.27 is suffices to show that (ii) implies (iii). So, suppose that $A^{\vee} \cap x^{\perp} = A^{\perp}$. Then, on use of 1.2.10, 1.1.5 and 1.4.6 we get

$$A - \operatorname{cone}(x) = A + \langle x \rangle = A^{\vee\vee} + x^{\perp\vee} = (A^{\vee} \cap x^{\perp})^{\vee} = A^{\perp\vee} = \langle A \rangle$$

and thus the claim. □

The facial relation behaves well on polycones, as we can see from the next results.

(1.4.8) Proposition *The relation $B \preccurlyeq A$ is an ordering on the set of polycones in V.*

Proof. The relation $B \preccurlyeq A$ is reflexive and antisymmetric, and so it remains to show that it is transitive. Let A, B and C be polycones in V such that $C \preccurlyeq B \preccurlyeq A$, and let X be a finite generating set of A. There are $u \in A^{\vee}$ and $v \in B^{\vee}$ such that $B = A \cap u^{\perp}$ and that $C = B \cap v^{\perp}$. As $\mathbb{R}$ is Archimedean, we know that for every $x \in X$ with $u(x) > 0$ there exists an $r_x \in \mathbb{N}$ such that $v(x) + r_x u(x) > 0$, and since X is finite there exists an $r \in \mathbb{N}$ with $r > \sup\{r_x \mid x \in X \wedge u(x) > 0\}$. Setting $w := v + ru \in V^*$, it is readily checked that $w \in A^{\vee}$ and that $C = A \cap w^{\perp}$, and thus the claim is proven. □

(1.4.9) The relation $B \preccurlyeq A$ is not necessarily transitive and hence not an ordering on the set of closed conic subsets of V. To give a counterexample in $\mathbb{R}^3$ we set $A := \operatorname{cone}(\{(x, y, 1) \in \mathbb{R}^3 \mid y \geq 0 \wedge x^2 + y^2 = 1\} \cup \{(-1, -1, 1)\})$, $B := \operatorname{cone}(\{(-1, 0, 1), (-1, -1, 1)\})$ and $C := \operatorname{cone}(\{(-1, 0, 1)\})$. Then, it is easy to check that $C \preccurlyeq B \preccurlyeq A$, but $C \not\preccurlyeq A$.

(1.4.10) Proposition *Let $A \subseteq V$ be a polycone. Then, it holds $s(A) \in$ face(A) and $s(A) = \bigcap$face(A), and it holds $0 \in$ face(A) if and only if A is sharp.*

Proof. Let X be a finite generating set of A, and moreover let $Y := \{x \in X \mid -x \notin A\}$. By 1.2.16, 1.2.13 b) and 1.2.26 there exists for every $x \in Y$ a $u_x \in A^{\vee}$ with $u_x(x) > 0$, and as X is finite we can set $u := \sum_{x \in Y} u_x \in A^{\vee}$. Clearly, it holds $s(A) \subseteq A \cap u^{\perp}$. Moreover, it is readily checked that $X \cap u^{\perp} \subseteq -A$ and hence $s(A) = A \cap u^{\perp}$. Therefore, $s(A)$ is a face of A, and hence the remaining claims follow easily from 1.3.5, 1.3.4 and 1.3.1. □

We have gathered enough knowledge about polycones to give an interpretation of the facial relation in terms of separability by hyperplanes. This will have different applications, a prominent one in the construction of completions (see 3.6.8), and a further one in proving separatedness of toric schemes (see 4.3.4 and IV.1.2.1). The proof given here is based on Botts's article [9].

(1.4.11) Lemma *Let A and B be conic subsets of V, and let f denote the canonical epimorphism $V \twoheadrightarrow V/\langle A \cap B\rangle$ in $\mathsf{Mod}(\mathbb{R})$. Then, it holds $f(A) \cap f(B) = 0$.*

Proof. Let $x \in f(A) \cap f(B)$. Then, there are $y \in A$ and $z \in B$ with $f(y) = x = f(z)$, and thus it follows $y - z \in \langle A \cap B\rangle = A \cap B - A \cap B$ by 1.2.10. Hence, there are $v, w \in A \cap B$ such that $y - z = v - w$, and this implies $y + w = z + v \in A \cap B$. Finally we get

$$x = f(y) = f(y + w) \in f(A \cap B) = 0$$

and therefore the claim. $\square$

(1.4.12) Lemma *Let A and B be sharp (finitely) W-conic subsets of V such that $A \cap B = 0$. Then, $A - B$ is a sharp (finitely) W-conic subset of V.*

Proof. We know from 1.2.7 that $A - B$ is (finitely) W-conic, and so it remains to show that is it sharp. Let $x \in s(A-B)$. Then, there are $y, y' \in A$ and $z, z' \in B$ with $y-z = x = z'-y'$, and it follows $y+y' = z+z' \in A\cap B = 0$, hence $y = -y' \in s(A)$ and $z = -z' \in s(B)$. Since A and B are sharp we get $x = y - z = 0$, and thus $A - B$ is sharp. $\square$

(1.4.13) Lemma *Let A and B be subsets of V with $A \cap B \neq \emptyset$ such that $U := \langle A \cap B\rangle$ is W-rational, and let $f : V \twoheadrightarrow V/U$ denote the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$. Moreover, suppose that $A \cap U = A \cap B = B \cap U$ and that there is a $v \in (W/U)^* \setminus 0$ such that $f(A - B) \subseteq v^\vee$ and that $f(A - B) \cap v^\perp = 0$. Then, A and B are W-separable in their intersection.*

Proof. As $A \cap B \neq \emptyset$ it holds $0 \in f(A) \cap f(B)$, and hence we have $f(A) \subseteq v^\vee$, $f(B) \subseteq (-v)^\vee$ and $f(A) \cap v^\perp = 0 = f(B) \cap v^\perp$. Setting $u := v \circ f \in V^* \setminus 0$ we get $A \subseteq u^\vee$ and $B \subseteq (-u)^\vee$. Furthermore, it holds $A \cap u^\perp = A \cap B = B \cap u^\perp$, and thus the linear hyperplane $u^\perp$ in V separates A and B in their intersection. Finally, as f is rational with respect to W and W/U it follows that $u \in W^*$, and hence $u^\perp$ is a linear W-hyperplane. $\square$

(1.4.14) Proposition *Let A and B be W-polycones in V such that $A \cap B$ is not full. Then, the following statements are equivalent:*

(i) A and B are W-separable in their intersection;

(ii) $A \cap B \in \operatorname{face}(A) \cap \operatorname{face}(B)$.

Proof. Obviously, (i) implies (ii). Conversely, we suppose that $A \cap B \in \operatorname{face}(A) \cap \operatorname{face}(B)$. If $A = B$, then there is a linear W-hyperplane in V containing $A \cap B$ and thus separating A and B in their intersection. So, suppose that $A \neq B$, set $U := \langle A \cap B\rangle$, and denote by $f : V \twoheadrightarrow V/U$ the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$. Clearly we have $0 \in A \cap B$, and 1.3.13 yields $A \cap U = A \cap B = B \cap U$. Moreover, 1.3.6 b) and 1.4.11 imply that $f(A)$ and $f(B)$ are sharp W/U-polycones in V/U with $f(A) \cap f(B) = 0$.

Therefore, $f(A-B)$ is a sharp W/U-polycone in V/U by 1.4.12, and by 1.4.10 it holds $0 \in \mathrm{face}(f(A-B))$, that is on use of 1.4.3, there is a $v \in (W/U)^*$ such that $f(A-B) \subseteq v^\vee$ and that $f(A-B) \cap v^\perp = 0$. If $v = 0$, then we get on use of 1.3.13 the contradiction $A = B$, and hence we have $v \neq 0$. Now, the claim follows on use of 1.4.13. □

If we consider the faces of a polycone A, then we can see that some of them play a more important role than others. A first example appeared in 1.4.10 concerning the summit $s(A)$. Now we consider the faces of almost minimal and almost maximal dimension, that is, the face of dimension $\dim(s(A)) + 1$ and $\dim(A) - 1$. In doing this we are lead amongst other things to a stronger version of 1.3.8, given in 1.4.21.

(1.4.15) Proposition *Let A be a sharp W-polycone in V, and let $X \subseteq W$. Then, the following statements are equivalent:*

(i) X is a minimal W-generating set of A;

(ii) The map $x \mapsto \mathrm{cone}(x)$ is a bijection from X to A_1.

Proof. Suppose that X is a minimal W-generating set of A, and let $l := \mathrm{Card}(X)$. On use of 1.3.3 it is seen that every 1-dimensional face of A has the form $\mathrm{cone}(x)$ for some $x \in X$. By minimality of X, to show (ii) it suffices to show that $\mathrm{cone}(x) \preccurlyeq A$ for every $x \in X$. This we do by induction on l. If $l \leq 1$, then this is obvious. So, let $l > 1$, suppose that the claim is true for W-polycones with a minimal generating set of cardinality strictly less than l, and let $x \in X$. We assume that $x \in \mathrm{in}_{\langle A\rangle}(A)$. By 1.4.7 it holds $A - A = A - \mathrm{cone}(x)$, and hence there is a family $(r_y)_{y\in X}$ in $\mathbb{R}_{\geq 0}$ and an $r \in \mathbb{R}_{\geq 0}$ such that

$$-\textstyle\sum_{y\in X\setminus\{x\}} y = \sum_{y\in X} r_y y - rx.$$

It follows

$$(r - r_x)x = \textstyle\sum_{y\in X\setminus\{x\}}(r_y + 1)y \in \mathrm{cone}(X \setminus \{x\}).$$

Minimality of X and sharpness of A imply $r = r_x$. Let $z \in X \setminus \{x\}$. Then, we get

$$-z = \textstyle\sum_{y\in X\setminus\{x,z\}} \frac{r_y+1}{r_z+1} y \in \mathrm{cone}(X \setminus \{x, z\}),$$

contradicting that A is sharp. Thus, we have $x \in \mathrm{fr}_{\langle A\rangle}(A)$. Now, by 1.3.8 there is a $B \prec A$ such that $x \in B$, and by 1.3.3 there is a proper subset $Y \subsetneqq X$ such that $B = \mathrm{cone}(Y)$. Minimality of X implies that Y is a minimal W-generating set of B of cardinality strictly less than X and contains x. This yields $\mathrm{cone}(x) \preccurlyeq B$, and therefore $\mathrm{cone}(x) \preccurlyeq A$ by 1.4.8 as desired.

Conversely, suppose that (ii) holds. There is a minimal W-generating set Y of A, and the above yields a bijection $f : Y \xrightarrow{\cong} X$ such that for every $y \in Y$ we have $\mathrm{cone}(y) = \mathrm{cone}(f(y))$ and hence $f(y) = r_y y$ for some $r_y \in \mathbb{R}_{>0}$. Therefore, X is a W-generating set of A, and minimality of Y implies minimality of X. □

(1.4.16) Let A be a sharp W-polycone in V, and let $X \subseteq V$ be a minimal generating set of A. Then, there is a family $(r_x)_{x\in X}$ in $\mathbb{R}_{>0}$ such that $\{r_x x \mid x \in X\}$ is a minimal W-generating set of A, as was seen in the proof of 1.4.15.

(1.4.17) Corollary *If A is a sharp polycone in V, then it holds $A = \sum_{B\in A_1} B$.*

Proof. By choosing a minimal generating set of A this follows from 1.4.15. □

(1.4.18) Proposition *Let A be a polycone in V, and let $B \preccurlyeq A$. Then, it holds* $\dim(B) + \dim(A^\vee \cap B^\perp) = n$.

Proof. Let X be a finite generating set of A, let $u \in A^\vee$ be such that $B = A \cap u^\perp$, and let $v \in B^\perp$. Then, for every $x \in X$ with $u(x) > 0$ there exists an $r_x \in \mathbb{N}$ such that $r_x u(x) + v(x) \geq 0$, and hence there exists an $r \in \mathbb{N}$ such that for every $x \in X$ with $u(x) > 0$ we have $ru(x) + v(x) \geq 0$. We set $w := ru + v$, and then it is readily checked that $w \in A^\vee \cap B^\perp$ and therefore $v \in \langle A^\vee \cap B^\perp\rangle$. But this implies $B^\perp = \langle A^\vee \cap B^\perp\rangle$, and thus the claim follows from 1.1.5. □

(1.4.19) Corollary *If A is polycone in V, then every face of A is the intersection of a family in $A_{\dim(A)-1}$.*

Proof. Without loss of generality we can assume that A is full, and then $A^\vee$ is sharp by 1.2.10. Let $B \preccurlyeq A$. Then, $B' := A^\vee \cap B^\perp$ is a face of $A^\vee$ by 1.3.12, and 1.4.17 yields the existence of a family $(C_i)_{i\in I}$ in $A_1^\vee$ with $B' = \sum_{i\in I} C_i$. On use of 1.3.12 and 1.1.5 we get

$$B = A \cap (B')^\perp = \bigcap_{i\in I} A \cap C_i^\perp$$

with $A \cap C_i^\perp \preccurlyeq A$ for every $i \in I$, and 1.4.18 implies that

$$\dim(A \cap C_i^\perp) = n - \dim(C_i) = \dim(A) - 1$$

for every $i \in I$. Thus, the claim is proven. □

(1.4.20) Corollary *Let A be a polycone in V, and let $B \in$ pface(A). Then, there exists a $C \in A_{\dim(A)-1}$ with $B \preccurlyeq C$.*

Proof. This is clear by 1.4.19. □

(1.4.21) Corollary *Let A be a polycone in V. Then, it holds*

$$\mathrm{fr}_{\langle A\rangle}(A) = \bigcup A_{\dim(A)-1}.$$

Proof. This follows immediately from 1.3.8 and 1.4.20. □

(1.4.22) This stronger version of 1.3.8 does not necessarily hold for closed conic sets that are not polycones. For example, if we denote by A the conic hull of the set $\{(x, y, 1) \mid x^2 + y^2 = 1\}$ in $\mathbb{R}^3$, then this is a closed, conic set of dimension 3, but all its faces are of dimension 0 or 1. Therefore, $\bigcup A_{\dim(A)-1}$ is empty, whereas $\mathrm{fr}_{\langle A\rangle}(A)$ is not.

The following somewhat technical result describes some sort of "subdivision" of sharp polycones and will be used when dealing with subdivisions of fans in 2.4.

(1.4.23) Proposition *Let A be a sharp polycone in V, and let $B \in A_1$. Then, it holds $A = \bigcup\{B + C \mid B \not\preccurlyeq C \preccurlyeq A\}$.*

Proof. Let $r := \mathrm{Card}(A_1) \in \mathbb{N}$, let $Q := \bigcup\{B + C \mid B \not\preccurlyeq C \preccurlyeq A\}$, and let $p \in B \setminus 0$. Clearly, it holds $Q \subseteq A$, and since A is sharp we have $0 \preccurlyeq A$ by 1.4.10 and hence $B \subseteq Q$. So, it remains to show that $A \setminus B \subseteq Q$, and we do this by induction on r. If $r = 1$, then we have $A = B$ and hence the claim is clear.

So, let $r > 1$, and suppose the claim to be true for strictly smaller values of r. As $0 \preccurlyeq A$, there is a linear hyperplane $H \subseteq V$ such that A lies on one side of H and that $H \cap A = 0$. It is easy to see that $A = \mathbb{R}_{\geq 0}((H + p) \cap A)$, and therefore it suffices to show that $(H + p) \cap A \setminus \{p\} \subseteq Q$. Let $q \in (H + p) \cap A \setminus \{p\}$, and let $G := p + \mathrm{cone}(q - p)$. Since $(H + p) \cap A$ is compact by 1.2.17 and $G \subseteq H + p$ is not compact, we have $G \not\subseteq A$. It follows from 1.2.12 that $G \cap \mathrm{fr}_{\langle A\rangle}(A) \neq \emptyset$, and 1.3.8 implies that there are an $F \in \mathrm{pface}(A)$ and a $q' \in F$ such that $q \in [\![p, q']\!]$. If $B \not\subseteq F$, then we have $q \in \mathrm{cone}(B \cup F) = B + F$ and hence $q \in Q$. Otherwise, F is a sharp polycone with $q \in F$ and with $B \in F_1$ such that $\mathrm{Card}(F_1) < r$, and thus our hypothesis implies that $q \in F = \bigcup\{B + C \mid B \not\preccurlyeq C \preccurlyeq A\} \subseteq Q$. Herewith, the claim is proven. □

(1.4.24) The hypothesis that A is sharp in 1.4.23 is necessary. Indeed, otherwise we have $B = s(A)$, resulting on use of 1.4.10 in the right hand side of the given equation being empty.

In the remainder of this section we are concerned with polycones having a generating set that fulfils some independence relation. The easiest notion is that of simplicial cone, and this will be used often in Chapter IV. Simplicial cones are a special case of regular cones (with respect to some rational structure). The notion of regularity is more subtle than simpliciality, and as we will see it depends on the structure of the subring $R \subseteq \mathbb{R}$ with respect to which our rational structure is defined.

(1.4.25) Let $A \subseteq V$ be conic. If A has a generating set X that is free over $\mathbb{R}$, then it follows from 1.4.15 and 1.4.16 that X is a minimal generating set of A and moreover that every minimal generating set of A is free over $\mathbb{R}$. If this condition is fulfilled, then A is called *simplicial*. If A is W-conic, then it has a W-generating set that is free over R if and only if it has a W_K-generating set that is free over K, and this is equivalent to A being simplicial. Obviously, simplicial W-conic subsets of V are W-polycones in V. It follows easily from 1.3.3 that faces of simplicial polycones are again simplicial. In particular, 1.4.10 implies that simplicial polycones are sharp.

(1.4.26) Example If $n \leq 2$, then every polycone in V is simplicial. On the other hand, the $\mathbb{Z}^3$-polycone in $\mathbb{R}^3$ generated by $(1, 1, 1)$, $(-1, 1, 1)$, $(-1, -1, 1)$ and $(1, -1, 1)$ is not simplicial.

(1.4.27) Lemma *Let $B \subseteq A \subseteq V$, and suppose that A is free. Then, it holds* $\operatorname{cone}(B) \preccurlyeq \operatorname{cone}(A)$.

Proof. Since A is free there exists $u \in V^*$ with $u(x) = 0$ for $x \in B$ and $u(x) \in \mathbb{R}_{>0}$ for $x \in A \setminus B$, and then 1.3.2 yields the claim. □

(1.4.28) If $A \subseteq V$ is not free and $B \subseteq A$, then $\operatorname{cone}(B)$ is not necessarily a face of $\operatorname{cone}(A)$. Indeed, a counterexample with $V = \mathbb{R}^3$ is given by $A = \{(1, 1, 1), (-1, 1, 1), (1, -1, 1), (-1, -1, 1)\}$ and $B = \{(-1, 1, 1), (1, -1, 1)\}$.

(1.4.29) Let A be a 1-dimensional sharp W-polycone in V. Note that A corresponds to a structure of oriented $\mathbb{R}$-vector space on $\langle A \rangle$. Hence, every $x \in A \setminus 0$ corresponds to an isomorphism of oriented $\mathbb{R}$-vector spaces[8] $u_x : \mathbb{R} \to \langle A \rangle$, $r \mapsto rx$. Clearly, if $x \in W \cap A \setminus 0$, then u_x is rational with respect to R and $W \cap \langle A \rangle$. Moreover, transport by means of u_x yields a structure of totally ordered $\mathbb{R}$-vector space on $\langle A \rangle$ that is independent of the choice of x. Indeed, it is easy to see that for $y, z \in A$ it holds $y \leq z$ if and only if $y \in [\![0, z]\!]$. This ordering on $\langle A \rangle$ is called *the canonical ordering on $\langle A \rangle$.*

An element $x \in V \setminus 0$ is called *W-minimal* if it is a minimal (and hence the smallest) element of $W \cap \operatorname{cone}(x) \setminus 0$ with respect to the ordering induced by the canonical ordering on $\langle x \rangle$.

If A is again a 1-dimensional sharp W-polycone in V, then the unique element of a minimal W-generating set of A is called a *W-generator of A.* If it is W-minimal, then it is uniquely determined by A, and if no confusion can arise it is denoted by A_W.

(1.4.30) A subset $A \subseteq V$ is called *W-regular* if it is contained in an R-basis of W (and in particular in W). By abuse of language, an element $x \in V$ is called *W-regular* if $\{x\}$ is W-regular. Clearly, subsets of W-regular sets are again W-regular.

If $R = K$, then $A \subseteq W$ is W-regular if and only if it is free over K, and this is the case if and only if it is free over $\mathbb{R}$. If A is W-regular, then it is free over R. Conversely, if $A \subseteq W$ is free over R, then it is free over K and hence W_K-regular by the above.

(1.4.31) Lemma *Let $x \in W$. We consider the following statements:*

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[8] where the canonical orientation on $\mathbb{R}$ is given by $\mathbb{R}_{>0}$.

(1) x is W-regular;

(2) It holds $\langle x\rangle_K \cap W = \langle x\rangle_R$ and $x \neq 0$;

(3) $\langle x\rangle_R$ is a direct factor of W and $x \neq 0$;

(4) x is indivisible in W.[9]

a) It holds (1)⇒(2)⇒(3)⇔(4).

b) If R is a principal ideal domain, then it holds (1)⇔(2)⇔(3)⇔(4).

Proof. a) First, suppose that there is an R-basis E of W with $x \in E$, and let $y \in \langle x\rangle_K \cap W$. Then, there is an $r \in K$ with $y = rx \in W$, and hence there is a family $(r_e)_{e\in E}$ in R with $rx = \sum_{e\in E} r_e e$. Since E is free over K it follows $rx = r_x x$, hence $r = r_x \in R$ and therefore $y \in \langle x\rangle_R$. The other inclusion and the condition $x \neq 0$ holding obviously, this proves that (1) implies (2).

Next, suppose that $\langle x\rangle_K \cap W = \langle x\rangle_R$. Since $\langle x\rangle_K$ has a W-rational complement U in W_K by 1.1.11, it follows from

$$W = (\langle x\rangle_K \cap W) \oplus (U \cap W) = \langle x\rangle_R \oplus (U \cap W)$$

that $\langle x\rangle_R$ is a direct factor or W. Hence, (2) implies (3).

Now, suppose that $\langle x\rangle_R$ is a direct factor of W, that U is a complement of $\langle x\rangle_R$ in W, and that $x \neq 0$. Then, x is an R-basis of $\langle x\rangle_R$, and hence there is a $u \in W^*$ with $u(x) = 1$ and $u\restriction_U = 0$. This shows that x is indivisible, and thus (3) implies (4).

Finally, suppose that x is indivisible in W, and let $u \in W^*$ with $u(x) = 1$. It is readily checked that $W \to W$, $y \mapsto u(y)x$ is a projector of W^* with image $\langle x\rangle_R$, and therefore $\langle x\rangle_R$ is a direct factor of W. Thus, (4) implies (3).

b) Since (4) implies (1) by [A, VII.4.3 Lemme 1], this follows from a). □

(1.4.32) Lemma *A totally ordered, integral ring S has an extremal element*[10] *if and only if $S \cong \mathbb{Z}$, and then the only such element is* 1.

Proof. Obviously, 1 is the only extremal element of $\mathbb{Z}$. Conversely, let p be an extremal element of S. We have $p \leq 1$, and if $p < 1$, then we get the contradiction $0 < p^2 < p$ since S is integral. Hence, it holds $p = 1$. Now, keeping in mind that totally ordered rings have characteristic 0 ([A, VI.2.1 Proposition 1]) we identify $\mathbb{Z}$ with the prime subring of S and assume that $S \neq \mathbb{Z}$. Then, there is an $r \in S_{>0}\setminus\mathbb{Z}$ and hence an $s \in \mathbb{N}$ with $s < r < s+1$, yielding the contradiction $0 < r - s < 1$. This proves $S \cong \mathbb{Z}$ and thus the claim. □

[9] Generalising [A, VII.4.2], an element $x \in W$ is called *indivisible (in W)* if the morphism $c_W(x) : W^* \to R$ is an epimorphism, or – equivalently – if there exists a $u \in W^*$ with $u(x) = 1$.

[10] An element of an ordered group G is called *extremal* if it is a minimal element of $G_{>0}$. An element of an ordered ring S is called *extremal* if it is an extremal element of the ordered group underlying S.

(1.4.33) Proposition *A 1-dimensional sharp W-polycone A in V has a W-minimal W-generator if and only if $R = \mathbb{Z}$.*

PROOF. By 1.4.29, the existence of a W-minimal W-generator of A is equivalent to the existence of an extremal element of $R_{>0}$, and hence the claim follows from 1.4.32. □

(1.4.34) Corollary *If $x \in W \setminus 0$ is W-minimal, then it is W-regular, and it holds $R = \mathbb{Z}$.*

PROOF. From 1.4.33 it is clear that $R = \mathbb{Z}$, and it is easy to see that W-minimality of x implies $\langle x \rangle_K \cap W = \langle x \rangle_R$. As $\mathbb{Z}$ is a principal ideal domain the claim follows on use of 1.4.31 b). □

(1.4.35) Let A be a W-polycone in V. By abuse of language, A is called *W-regular* if it has a W-regular W-generating set. From 1.3.3 and 1.4.30 it is clear that faces of W-regular W-polycones are again W-regular. Moreover, 1.4.30 implies that W-regular W-polycones are simplicial and that simplicial W-polycones are W_K-regular. Hence, in case $R = K$, the properties of a W-polycone to be W-regular, simplicial, or V-regular, are equivalent.

(1.4.36) Example Let $R = \mathbb{Z}$ and let A be a sharp W-polycone in V with $\dim(A) \leq 1$. Then, A is W-regular. Indeed, if $\dim(A) = 0$ this is trivial, and if $\dim(A) = 1$, then the W-minimal W-generator A_W of A is W-regular by 1.4.34.

(1.4.37) Example The $\mathbb{Z}^2$-polycone in $\mathbb{R}^2$ generated by $(1,1)$ and $(-1,1)$ is $\mathbb{Q}^2$-regular (that is, simplicial), but not $\mathbb{Z}^2$-regular.

1.5. Direct sums and decompositions of polycones

Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, and let W be an R-structure on V.

It may seem naive to define a direct sum of polycones to be a sum of polycones such that the sum of the vector spaces generated by these polycones is direct. But it will get clear rather soon that this idea works quite well.

(1.5.1) Let $(A_i)_{i \in I}$ be a finite family of W-conic subsets of V. For every $i \in I$ we set $V_i := \langle A_i \rangle$, and we denote by W_i the R-structure induced by W on V_i. Moreover, we set $V' := \bigoplus_{i \in I} V_i$ and $W' := \bigoplus_{i \in I} W_i$, keeping in mind that W' is an R-structure on V', that the canonical injection $\iota_i : V_i \rightarrowtail V'$ in $\mathsf{Mod}(\mathbb{R})$ is rational with respect to W_i and W' for every $i \in I$, and that the canonical morphism

$$\varepsilon : V' \to V,\ (x_i)_{i \in I} \mapsto \sum_{i \in I} x_i$$

is rational with respect to W' and W (1.1.11). By restriction and coastriction, ε induces a surjection $\varepsilon' : \sum_{i\in I} \iota_i(A_i) \to \sum_{i\in I} A_i$.

The map ε is injective, that is, the sum of $\mathbb{R}$-vector spaces $\sum_{i\in I} V_i$ is direct, if and only if the map ε' is injective. Indeed, it is clear that injectivity of ε implies injectivity of ε'. Conversely, suppose that ε' is injective, and let $(a_i)_{i\in I} \in \mathrm{Ker}(\varepsilon)$. It follows from 1.2.10 that for every $i \in I$ there exist $a_i', a_i'' \in A_i$ with $a_i = a_i' - a_i''$. This implies

$$\varepsilon'((a_i')_{i\in I}) = \sum_{i\in I} a_i' = \sum_{i\in I} a_i'' = \varepsilon'((a_i'')_{i\in I}),$$

hence $a_i' = a_i''$ for every $i \in I$ and therefore $(a_i)_{i\in I} = 0$. Thus, ε is injective.

Using the above we see that the map ε' is injective if and only if for every $x \in \sum_{i\in I} A_i$ there exists for every $i \in I$ a unique $x_i \in A_i$ such that $x = \sum_{i\in I} x_i$.

By abuse of language, we denote $\sum_{i\in I} \iota_i(A_i)$ by $\bigoplus_{i\in I} A_i$ and call it *the direct sum of (the family of conic sets)* $(A_i)_{i\in I}$. Moreover, we say that *the sum of (the family of conic sets)* $(A_i)_{i\in I}$ *is direct* if the map ε' is injective (and hence bijective). If this is the case, then we identify $\bigoplus_{i\in I} V_i$ and $\sum_{i\in I} V_i$, and hence $\bigoplus_{i\in I} A_i$ and $\sum_{i\in I} A_i$, by means of ε and ε' respectively.

It is clear from the above that

$$\dim\Bigl(\bigoplus_{i\in I} A_i\Bigr) = \sum_{i\in I} \dim(A_i).$$

Finally, if A_i is a W-polycone for every $i \in I$, then $\bigoplus_{i\in I} A_i$ is a W'-polycone in V'.

(1.5.2) Lemma *Let $(V_i)_{i\in I}$ be a finite family of $\mathbb{R}$-vector spaces of finite dimension with $V := \bigoplus_{i\in I} V_i$. Moreover, for every $i \in I$ let $u_i \in V_i^*$ and $A_i \subseteq V_i$ such that $0 \in A_i$.*

a) It holds $\sum_{i\in I} A_i \subseteq (\sum_{i\in I} u_i)^{\vee,V}$ if and only if $A_i \subseteq u_i^{\vee,V_i}$ for every $i \in I$.

b) If $\sum_{i\in I} A_i \subseteq (\sum_{i\in I} u_i)^{\vee,V}$, then it holds

$$\sum_{i\in I}(A_i \cap u_i^{\perp,V_i}) = \Bigl(\sum_{i\in I} A_i\Bigr) \cap \Bigl(\sum_{i\in I} u_i\Bigr)^{\perp,V}.$$

Proof. Straightforward. □

(1.5.3) Proposition *Let $(A_i)_{i\in I}$ be a finite family of conic subsets of V such that the sum $A := \sum_{i\in I} A_i$ is direct.*

a) It holds

$$\mathrm{face}(A) = \Bigl\{\bigoplus_{i\in I} B_i \;\Big|\; \forall i \in I : B_i \preccurlyeq A_i\Bigr\}.$$

b) If $B_i \preccurlyeq A_i$ for every $i \in I$, then it holds $B_i = (\bigoplus_{j\in I} B_j) \cap A_i$ for every $i \in I$.

c) If $B \preccurlyeq A$, then it holds $B = \bigoplus_{i\in I} B \cap A_i$ and $B \cap A_i \preccurlyeq A_i$ for every $i \in I$.

Proof. For every $i \in I$ we set $V_i := \langle A_i \rangle$.

a) Let $B \subseteq V$. It holds $B \preccurlyeq A$ if and only if there is a $u \in V^*$ such that $A \subseteq u^{\vee,V}$ and that $B = A \cap u^{\perp,V}$. Since $V^* = \bigoplus_{i\in I} V_i^*$ this holds if and only if there exists for every $i \in I$ a $u_i \in V_i^*$ such that $A \subseteq (\sum_{i\in I} u_i)^{\vee,V}$ and that $B = A \cap (\sum_{i\in I} u_i)^{\perp,V}$. By 1.5.2, this is the case if and only if there exists for every $i \in I$ a $u_i \in V_i^*$ such that $A_i \subseteq u_i^{\vee,V_i}$ and that $B = \sum_{i\in I} A_i \cap u_i^{\perp,V_i}$. Finally, this holds if and only if there exists for every $i \in I$ a $B_i \preccurlyeq A_i$ such that $B = \sum_{i\in I} B_i$, and as this sum is obviously direct claim a) is proven.

b) Let $i \in I$, and let $x \in (\bigoplus_{j\in I} B_j) \cap A_i$. Then, for every $j \in I$ there is an $x_j \in B_j$ with $x = \sum_{j\in I} x_j$, and since $x, x_i \in V_i$ it follows

$$\textstyle\sum_{j\in I\setminus\{i\}} x_j \in (\sum_{j\in I\setminus\{i\}} V_j) \cap V_i = 0.$$

This implies $x = x_i \in B_i$. The other inclusion being obvious, this shows the claim.

c) By a) there exists for every $i \in I$ a $B_i \preccurlyeq A_i$ such that $B = \bigoplus_{i\in I} B_i$, and b) yields $B \cap A_i = B_i \preccurlyeq A_i$ and thus the claim. □

(1.5.4) Corollary *Let $(A_i)_{i\in I}$ be a finite family of polycones in V with* $\mathrm{Card}(I) \neq 1$ *such that the sum $A := \sum_{i\in I} A_i$ is direct. Then, the following statements are equivalent:*

(i) A is sharp;

(ii) A_i is sharp for every $i \in I$;

(iii) $A_i \preccurlyeq A$ for every $i \in I$.

Proof. This follows easily on use of 1.4.10, 1.5.3 and 1.4.8. □

A notion of direct sum of some objects leads obviously to a notion of decomposition into indecomposable such objects, and to the question of existence and uniqueness of such a decomposition. As we will see, in case of direct sums of polycones the answers are positive concerning existence in general, and concerning uniqueness for sharp polycones.[11]

(1.5.5) A polycone A in V is called *W-decomposable* if it is the direct sum of two W-polycones in V that are different from 0, and *W-indecomposable* otherwise.

If A is a polycone in V, then a *W-decomposition of A* is a set Z of W-indecomposable W-polycones in V that are different from 0 such that $A = \bigoplus Z$. If Z is a W-decomposition of A, then it clearly holds $\mathrm{Card}(Z) \leq \dim(A)$. Moreover, if Z is a W-decomposition of A, then A is a W-polycone.

(1.5.6) Example Every W-polycone A in V with $\dim(A) \leq 1$ is W-indecomposable.

[11]The notion of decomposition of polycones used here is not to be confused with the one coming from Minkowski sums as treated for example in [21].

(1.5.7) Example The $\mathbb{Z}^3$-polycone in $\mathbb{R}^3$ defined in 1.4.26 that is not simplicial, is $\mathbb{Z}^3$-indecomposable. The $\mathbb{Z}^4$-polycone in $\mathbb{R}^4$ generated by

$$\{(1,1,1,0),(-1,1,1,0),(1,-1,1,0),(-1,-1,1,0),(0,0,0,1)\}$$

has a $\mathbb{Z}^4$-decomposition consisting of the $\mathbb{Z}^4$-polycone generated by

$$\{(1,1,1,0),(-1,1,1,0),(1,-1,1,0),(-1,-1,1,0)\},$$

and the $\mathbb{Z}^4$-polycone generated by $(0,0,0,1)$.

(1.5.8) A W-polycone A in V is simplicial if and only if it is sharp and A_1 is a W-decomposition of A. Indeed, this is clear from 1.4.17 and 1.4.25. Moreover, this yields that if A is a simplicial W-polycone in V, then A is W-indecomposable if and only if $\dim(A) \leq 1$.

Now, let $(A_i)_{i\in I}$ be a finite family of polycones in V. Then, $A := \bigoplus_{i\in I} A_i$ is simplicial if and only if A_i is simplicial for every $i \in I$. Indeed, if A is simplicial, then it is sharp by 1.4.25, and hence $A_i \preccurlyeq A$ is simplicial for every $i \in I$ by 1.5.4 and 1.4.25. The converse follows easily from what was shown above and 1.5.4.

(1.5.9) Lemma *Let A be a sharp W-polycone in V, let Z be a W-decomposition of A, and let $u_F \in F \setminus 0$ for every $F \in A_1$. Moreover, let $U := \{u_F \mid F \in A_1\}$, and for every $P \in Z$ let $U_P := \{u_F \mid F \in P_1\}$.*

a) $(U_P)_{P\in Z}$ is a partition[12] *of U.*

b) The sum of $\mathbb{R}$-vector spaces $\sum_{P\in Z}\langle U_P\rangle$ is direct.

c) If $P \in Z$, then every partition $(C_l)_{l\in L}$ of U_P such that the sum of $\mathbb{R}$-vector spaces $\sum_{l\in L}\langle C_l\rangle$ is direct, has cardinality 1.

Proof. For $F \in A_1$ there exists by 1.5.3 a) a $P \in Z$ such that $F \in P_1$, and hence we have $U \subseteq \bigcup_{P\in Z} U_P$. For $P \in Z$ and every $F \in P_1$ it follows from 1.5.4 that $P \preccurlyeq A$, hence $F \in A_1$ by 1.4.8 and thus $\bigcup_{P\in Z} U_P \subseteq U$. For $P, P' \in Z$ and $F \in P_1 \cap P'_1$ it holds $P \cap P' \neq 0$ and hence $P = P'$, and for $P \in Z$ we have $P \neq 0$ and hence $P_1 \neq \emptyset$ by 1.4.15. Thus, $(P_1)_{P\in Z}$ is a partition of A_1, and therefore $(U_P)_{P\in Z}$ is a partition of U. Hence, a) is proven.

We have $\sum_{P\in Z}\langle U_P\rangle = \sum_{P\in Z}\langle P\rangle$ by 1.4.15, and this sum is direct, for Z is a W-decomposition of A. Thus, b) is proven. Finally, let $P \in Z$, and assume that there is a partition (G, H) of U_P with $\langle G\rangle \cap \langle H\rangle = 0$. Then, $\operatorname{cone}(G)$ and $\operatorname{cone}(H)$ are W-polycones different from 0, and it holds $P = \operatorname{cone}(G) \oplus \operatorname{cone}(H)$, contradictory to P being W-indecomposable. Now, c) follows from this. □

(1.5.10) Lemma *Let $U \subseteq V \setminus 0$ be finite. Then, there is at most one partition $(A_p)_{p\in P}$ of U with the following properties:*

[12] If E is a set, then a *partition of* E is a set P of pairwise disjoint, nonempty subsets of E such that $E = \bigcup P$, and then, by abuse of language, we also call the family Id_P a partition of E.

i) The sum of $\mathbb{R}$*-vector spaces* $\sum_{p\in P}\langle A_p\rangle$ *is direct;*

ii) If $p \in P$*, then every partition* $(C_l)_{l\in L}$ *of* A_p *such that the sum of* $\mathbb{R}$*-vector spaces* $\sum_{l\in L}\langle C_l\rangle$ *is direct, has cardinality* 1*.*

Proof. Let $(A_p)_{p\in P}$ and $(B_q)_{q\in Q}$ be partitions of U having the above properties i) and ii). Let $q \in Q$. For every $u \in B_q$ there is a unique $p_u \in P$ with $u \in A_{p_u}$. So, $\{B_q \cap A_{p_u} \mid u \in B_q\}$ is a partition of B_q, and the sum $\sum\{\langle B_q \cap A_{p_u}\rangle \mid u \in B_q\}$ is direct by property i). By property ii) it follows that $\{B_q \cap A_{p_u} \mid u \in B_q\}$ has cardinality 1, and thus there is a unique $p \in P$ with $B_q \subseteq A_p$. Now, by reasons of symmetry we get $\{A_p \mid p \in P\} = \{B_q \mid q \in Q\}$ and hence the claim. $\square$

(1.5.11) Theorem *Every* W*-polycone in* V *has a* W*-decomposition, and every sharp* W*-polycone in* V *has a unique* W*-decomposition.*

Proof. It is clear that every W-indecomposable W-polycone in V has a W-decomposition. Let A be a W-polycone in V. We show the existence of a W-decomposition of A by induction on $d := \dim(A)$. If $d \leq 1$, then A is W-indecomposable by 1.5.6 and hence has a W-decomposition. So, let $d > 1$, and suppose that every W-polycone in V of dimension strictly smaller than d has a W-decomposition. If A is W-indecomposable, then it has a W-decomposition. Otherwise, there are W-polycones B and C in V, different from 0, such that $A = B \oplus C$. Since $d = \dim(B)+\dim(C)$ it follows that B and C both have W-decompositions, and it is clear that the union of these W-decompositions is a W-decomposition of A. Thus, the first claim is proven.

Now, let A be a sharp W-polycone in V, and let Z and Z' be W-decompositions of A. For every $F \in A_1$ let $u_F \in F \cap W \setminus 0$, and let $U := \{u_F \mid F \in A_1\}$. Then, by 1.5.9 we have the partitions $(\{u_F \mid F \in P_1\})_{P\in Z}$ and $(\{u_F \mid F \in Q_1\})_{Q\in Z'}$ of U, and they fulfil the hypothesis of 1.5.10. Hence, they coincide, and thus we get

$$Z = \{\mathrm{cone}(\{u_F \mid F \in P_1\}) \mid P \in Z\} = \{\mathrm{cone}(\{u_F \mid F \in Q_1\}) \mid Q \in Z'\} = Z'. \qquad \square$$

(1.5.12) The uniqueness statement in 1.5.11 cannot be extended to not necessarily sharp polycones, since decompositions of sub-$\mathbb{R}$-vector spaces of V of dimension at least 2 are not unique.

(1.5.13) Lemma *Let* A*,* B *and* C *be conic subsets of* V *with* $\dim(A) = 1$ *and* $A \not\subseteq B \preccurlyeq C \supseteq A$*. Then, the sum* $A + B$ *is direct.*

Proof. There is a $u \in C^\vee$ such that $B = C \cap u^\perp$ and hence $\langle B\rangle \subseteq u^\perp$. If $\langle A\rangle \cap \langle B\rangle \neq 0$, then this $\mathbb{R}$-vector space contains a line, necessarily equal to $\langle A\rangle$, and from this we get the contradiction $A \subseteq C \cap \langle A\rangle \subseteq C \cap \langle B\rangle \subseteq C \cap u^\perp = B$. $\square$

2. Fans

Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, and let W be an R-structure on V.

2.1. Semifans

Now it is time for the main combinatorial objects of the whole theory to show up: fans. We consider first the slightly more general semifans as they naturally appear sometimes, for example in 3.4.1.

(2.1.1) A *W-semifan (in V)* is a finite set Σ of W-polycones in V such that for all $\sigma, \tau \in \Sigma$ it holds $\sigma \cap \tau \in \text{face}(\sigma) \subseteq \Sigma$.[13] If Σ is a W-semifan in V, then an element of Σ is called a *cone in* Σ. In case $W = V$ we speak just of semifans in V.

(2.1.2) Example If σ is a W-polycone in V, then face(σ) is a W-semifan in V called *the facial semifan of* σ.

(2.1.3) Let Σ be a semifan in V. The relation $\sigma \preccurlyeq \tau$ induces an ordering on Σ, coinciding with the ordering $\sigma \subseteq \tau$. By means of this we consider Σ as an ordered set. Then, Σ is a lower semilattice, and for a family $(\sigma_i)_{i \in I}$ in Σ (considered as a family of polycones in V) it holds $\inf\{\sigma_i \mid i \in I\} = \bigcap_{i \in I} \sigma_i$. We set $s(\Sigma) := \bigcap \Sigma$ and call this *the summit of* Σ. If Σ is nonempty, then $s(\Sigma)$ is the smallest element of Σ.

It holds $s(\sigma) = s(\tau)$ for all cones σ and τ in Σ, and hence it holds $s(\sigma) = s(\Sigma)$ for every cone σ in Σ. Indeed, $\sigma \cap \tau$ is a face of σ and of τ, and as $s(\sigma)$ is the smallest face of σ by 1.4.10 we get $s(\sigma) \preccurlyeq \sigma \cap \tau$ and hence $s(\sigma) \preccurlyeq \tau$. But as $s(\tau)$ is the smallest face of τ, it follows $s(\tau) \preccurlyeq s(\sigma)$, and the claim follows from this by reasons of symmetry.

In particular, if Σ is a W-semifan in V, then $s(\Sigma)$ is a W-rational sub-$\mathbb{R}$-vector space of V.

(2.1.4) Let Σ be a semifan in V. For $k \in \mathbb{N}_0$ we set

$$\Sigma_k := \{\sigma \in \Sigma \mid \dim(\sigma) = k\},$$

and we define analogously the sets $\Sigma_{\leq k}$, $\Sigma_{<k}$, $\Sigma_{\geq k}$ and $\Sigma_{>k}$, and furthermore we set $\Sigma(k) := \bigcup_{\sigma \in \Sigma_k} \text{face}(\sigma)$. Moreover, we denote by $\Sigma_{\max}$ the set of maximal elements of Σ, and we set[14] $\mathfrak{D}(\Sigma) := \Sigma_{\max} \setminus \Sigma_n$ and

$$\mathfrak{F}(\Sigma) := \{\sigma \in \Sigma_{n-1} \mid \exists! \tau \in \Sigma_n : \sigma \preccurlyeq \tau\}.$$

If $\sigma \in \Sigma$, then we set $\Sigma_\sigma := \{\tau \in \Sigma \mid \sigma \preccurlyeq \tau\}$, and obviously we have $\bigcap \Sigma_\sigma = \sigma$.

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[13]This definition may seem asymmetrical in σ and τ, but it is not.

[14]Note that the sets $\mathfrak{D}(\Sigma)$ and $\mathfrak{F}(\Sigma)$ depend on V.

The semifan Σ is called *fulldimensional (in V)* if $\Sigma_n \neq \emptyset$, *equidimensional* if there is a $k \in \mathbb{N}_0$ such that $\Sigma_{\max} = \Sigma_k$, and *equifulldimensional (in V)* if it is equidimensional and fulldimensional in V.

(2.1.5) Let Σ be a semifan in V. The set $|\Sigma| := \bigcup \Sigma$ is called *the support of* Σ. By abuse of language, $\operatorname{cone}(\Sigma) := \operatorname{cone}(|\Sigma|)$ is called *the cone generated by* Σ, and $\langle \Sigma \rangle := \langle |\Sigma| \rangle$ is called *the space generated by* Σ. Furthermore, $\dim(\Sigma) := \dim(\langle \Sigma \rangle)$ is called *the dimension of* Σ. If Σ is a W-semifan in V, then $\operatorname{cone}(\Sigma)$ is a W-polycone in V, and $\langle \Sigma \rangle$ is a W-rational sub-$\mathbb{R}$-vector space of V.

A semifan Σ in V is called *complete (in V)* if $|\Sigma| = V$, *skeletal complete (in V)* if $\operatorname{cone}(\Sigma) = V$, and *full (in V)* if $\langle \Sigma \rangle = V$.

(2.1.6) Let Σ and Σ' be semifans in V such that $\Sigma \subseteq \Sigma'$. Then, Σ is called a *subsemifan of* Σ', and Σ' is called an *extension semifan of* Σ. If Σ' is a W-semifan, then so is Σ, and then Σ is called a *W-subsemifan of* Σ' and Σ' is called a *W-extension semifan of* Σ or, if no confusion can arise, a *W-extension of* Σ. A complete extension of Σ, or a complete W-extension of Σ, is called a *completion of* Σ *(in V)*, or a *W-completion of* Σ *(in V)*, respectively. If no confusion can arise, then we use expressions like "Let $\Sigma \subseteq \Sigma'$ be a W-extension (of semifans)", meaning that Σ is a W-semifan and Σ' is a W-extension of Σ.

(2.1.7) Example Let Σ be a semifan in V, and let $\mathrm{T} \subseteq \Sigma$. Then, $\Sigma' := \bigcup_{\sigma \in \mathrm{T}} \operatorname{face}(\sigma)$ is a subsemifan of Σ with $\Sigma'_{\max} \subseteq \mathrm{T} \subseteq \Sigma'$.

The sets $\mathfrak{D}(\Sigma)$ and $\mathfrak{F}(\Sigma)$ defined by a semifan Σ as introduced above will play a central role in the description of the topological frontier of $|\Sigma|$ – in fact, they *are* the frontier, as may be guessed by considering some simple sketches. The remaining results in this section aim at enlarging our knowledge about these two sets.

(2.1.8) If Σ is a semifan in V, then the subsemifan $\bigcup_{\sigma \in \mathfrak{F}(\Sigma)} \operatorname{face}(\sigma)$ of Σ is denoted by $\overline{\mathfrak{F}}(\Sigma)$.

(2.1.9) Let Σ be a semifan in V. For every $\sigma \in \Sigma$ there is a $\tau \in \Sigma_{\max}$ with $\sigma \preccurlyeq \tau$, and hence it holds $|\Sigma| = \bigcup \Sigma_{\max}$. Moreover, we have $\Sigma_n \subseteq \Sigma_{\max}$, and it holds $\Sigma_n = \Sigma_{\max}$ if and only if Σ is empty or equifulldimensional. If Σ is not fulldimensional, then $\mathfrak{F}(\Sigma)$ is empty.

(2.1.10) Let Σ be a semifan in V, and let $k \in \mathbb{N}_0$. Then, $\Sigma(k)$ is a subsemifan of Σ with $\Sigma(k)_{\max} = \Sigma_k$. In particular, $\Sigma(n)$ is equifulldimensional if and only if it is nonempty, and this is the case if and only if Σ is fulldimensional. Moreover, we clearly have $\mathfrak{F}(\Sigma) = \mathfrak{F}(\Sigma(n))$.

(2.1.11) Proposition *Let Σ, Σ' and Σ'' be semifans in V such that $\Sigma = \Sigma' \cup \Sigma''$.*

a) It holds

$$\Sigma_{\max} = (\Sigma'_{\max} \setminus \Sigma'') \amalg (\Sigma'_{\max} \cap \Sigma''_{\max}) \amalg (\Sigma''_{\max} \setminus \Sigma').$$

b) It holds

$$\mathfrak{D}(\Sigma) = (\mathfrak{D}(\Sigma') \setminus \Sigma'') \amalg (\mathfrak{D}(\Sigma') \cap \mathfrak{D}(\Sigma'')) \amalg (\mathfrak{D}(\Sigma'') \setminus \Sigma').$$

c) It holds

$$\mathfrak{F}(\Sigma) = \{\sigma \in \mathfrak{F}(\Sigma') \mid \Sigma''_\sigma \subseteq \Sigma'_\sigma\} \cup \{\sigma \in \mathfrak{F}(\Sigma'') \mid \Sigma'_\sigma \subseteq \Sigma''_\sigma\}.$$

Proof. a) is straightforward, and b) follows immediately from a). Let $\sigma \in \mathfrak{F}(\Sigma')$ be such that $\Sigma''_\sigma \subseteq \Sigma'_\sigma$. Then, it holds $\dim(\sigma) = n-1$, and there exists a unique $\tau \in \Sigma'_n$ with $\sigma \preccurlyeq \tau$. Moreover, $\Sigma''_\sigma \subseteq \Sigma'_\sigma = \{\sigma, \tau\}$ implies that there is at most one $\omega \in \Sigma''_n$ with $\sigma \preccurlyeq \omega$, namely τ. Hence, τ is the only cone in Σ_n containing σ, and therefore it holds $\sigma \in \mathfrak{F}(\Sigma)$. By reasons of symmetry, this proves the inclusion "$\supseteq$" in c).

Conversely, let $\sigma \in \mathfrak{F}(\Sigma)$. Then, it holds $\dim(\sigma) = n-1$, and there exists a unique $\tau \in \Sigma_n$ with $\sigma \preccurlyeq \tau$. If $\tau \in \Sigma' \setminus \Sigma''$, then it holds $\sigma \in \mathfrak{F}(\Sigma')$ and $\Sigma''_\sigma \subseteq \{\sigma\} \subseteq \{\sigma, \tau\} = \Sigma'_\sigma$. If $\tau \in \Sigma' \cap \Sigma''$, then it holds $\sigma \in \mathfrak{F}(\Sigma')$ and $\Sigma''_\sigma = \{\sigma, \tau\} = \Sigma'_\sigma$. Hence, by reasons of symmetry this proves the inclusion "$\subseteq$" in c). □

(2.1.12) Proposition *Let Σ be a semifan in V, and let $\sigma \in \Sigma$ such that $\Sigma_\sigma \cap \Sigma_n \neq \emptyset$.*

a) It holds

$$\sigma = \bigcap_{\tau \in \Sigma_\sigma \cap \Sigma_n} \bigcap \{\omega \in \Sigma_{n-1} \mid \sigma \preccurlyeq \omega \preccurlyeq \tau\}.$$

b) Let $\rho \in \Sigma$ be such that $\rho \not\preccurlyeq \sigma$ and that $\Sigma_n \subseteq \Sigma_\rho$. Then, there exists a $\tau \in \mathfrak{F}(\Sigma)$ with $\sigma \preccurlyeq \tau$.

Proof. a) For every $\tau \in \Sigma_\sigma \cap \Sigma_n$ it holds

$$\sigma = \bigcap \{\omega \in \Sigma_{n-1} \mid \sigma \preccurlyeq \omega \preccurlyeq \tau\}$$

by 1.4.19, and therefore $\Sigma_\sigma \cap \Sigma_n \neq \emptyset$ implies the claim.

b) We assume that the claim is not true. Then, it follows that for every $\tau \in \Sigma_{n-1} \cap \Sigma_\sigma$ there are $\omega, \omega' \in \Sigma_n$ with $\omega \neq \omega'$ such that $\tau \preccurlyeq \omega$ and $\tau \preccurlyeq \omega'$, hence $\tau = \omega \cap \omega'$ and $\rho \preccurlyeq \tau$. For every $\omega \in \Sigma_\sigma \cap \Sigma_n$ this implies

$$\rho \preccurlyeq \bigcap \{\tau \in \Sigma_{n-1} \mid \sigma \preccurlyeq \tau \preccurlyeq \omega\}$$

and hence on use of a) the contradiction

$$\rho \preccurlyeq \bigcap_{\omega \in \Sigma_\sigma \cap \Sigma_n} \bigcap \{\tau \in \Sigma_{n-1} \mid \sigma \preccurlyeq \tau \preccurlyeq \omega\} = \sigma.$$

□

2.2. Fans

Semifans consisting only of sharp cones are called fans. These will be used in Chapter IV to define toric schemes.

(2.2.1) A *W-fan (in V)* is a W-semifan Σ in V such that $s(\Sigma) = 0$ or $\Sigma = \emptyset$. In case $W = V$ we speak just of fans in V. If Σ is a semifan in V, then it clear from 2.1.3 and 1.4.10 that Σ is a fan if and only if every cone in Σ is sharp, and this is the case if and only if it is empty or if there is a cone in Σ that is sharp.

(2.2.2) Example If σ is a W-polycone in V, then its facial semifan face(σ) is a W-fan in V if and only if σ is sharp, and then it is called *the facial fan of σ.*

(2.2.3) Let $\mathbb{A}$ be a finite set of W-polycones in V. Then, it is readily checked that $\Sigma := \bigcup_{\sigma \in \mathbb{A}} \text{face}(\sigma)$ is a W-semifan in V if and only if for all $\sigma, \tau \in \mathbb{A}$ it holds $\sigma \cap \tau \preccurlyeq \sigma$. Moreover, if Σ is a semifan, then it is a fan if and only if every cone in $\mathbb{A}$ is sharp, or – equivalently by 2.2.1 – if $\mathbb{A}$ is empty or contains a sharp cone.

(2.2.4) Let Σ be a semifan in V and let Σ' be a W-extension semifan of Σ. If Σ' is a fan, then so is Σ, and if Σ is a nonempty fan, then so is Σ'. In these cases we speak of subfans and W-extension fans.

Now, we consider a semifan Σ, a cone σ in Σ, and the "image" of Σ under the canonical projection $V \twoheadrightarrow V/\langle\sigma\rangle$. We show that in this way we get a fan in the quotient space $V/\langle\sigma\rangle$ that shares a lot of properties with Σ "around σ". This technique allows to consider fans instead of semifans (see 2.2.8 and 2.2.10). Moreover, it provides the basis for arguments by induction on dimension, and in this form it will play a central role in the combinatorial description of the frontier of a semifan in Section 2.3, and also in the construction of completions in Section 3.

(2.2.5) Let Σ be a W-semifan in V, and let $\sigma \in \Sigma$. We set $V_\sigma := V/\langle\sigma\rangle$ and $W_\sigma := W/\langle\sigma\rangle$, and we denote by $p_\sigma : V \twoheadrightarrow V_\sigma$ the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$. The set

$$\Sigma/\sigma := \{p_\sigma(\tau) \mid \tau \in \Sigma_\sigma\}$$

is called *the projection of Σ along σ,* and we denote by

$$\widehat{p}_\sigma : \Sigma_\sigma \to \Sigma/\sigma,\ \tau \mapsto p_\sigma(\tau)$$

the surjection induced by p_σ. For $\tau \in \Sigma_\sigma$, we also write by abuse of language τ/σ instead of $p_\sigma(\tau)$.

(2.2.6) Proposition *Let Σ be a W-semifan in V, and let $\sigma \in \Sigma$. Then, Σ/σ is a W_σ-fan in V_σ, and $\widehat{p}_\sigma : \Sigma_\sigma \to \Sigma/\sigma$ is an isomorphism of ordered sets.*

Proof. From 1.2.5 we know that Σ/σ is a finite set of W_σ-polycones in V_σ. For $\tau \in \Sigma_\sigma$ it holds

$$\mathrm{face}(\tau/\sigma) = \{\omega/\sigma \mid \omega \in \Sigma_\sigma \wedge \omega \preccurlyeq \tau\} \subseteq \Sigma/\sigma$$

by 1.3.7 a), and this implies on use of 1.3.7 b) that for $\tau, \tau' \in \Sigma_\sigma$ it holds $(\tau/\sigma) \cap (\tau'/\sigma) = (\tau \cap \tau')/\sigma \preccurlyeq \tau/\sigma$. Thus, Σ/σ is a W_σ-semifan in V_σ. Again using 1.3.7 b) and keeping in mind 2.1.4 we see that

$$s(\Sigma/\sigma) = \bigcap \widehat{p}_\sigma(\Sigma_\sigma) = (\bigcap \Sigma_\sigma)/\sigma = \sigma/\sigma = 0,$$

and therefore Σ/σ is a W_σ-fan in V_σ.

The map $\widehat{p}_\sigma$ is surjective by definition, and it is obviously increasing. If $\tau, \omega \in \Sigma_\sigma$ are such that $\tau/\sigma = \omega/\sigma$, then 1.3.7 b) implies that $\tau/\sigma = (\tau \cap \omega)/\sigma$, and then it follows from 1.3.7 c) that $\tau = \tau \cap \omega \preccurlyeq \omega$. By reasons of symmetry we get that $\widehat{p}_\sigma$ is injective and hence bijective. Finally, let $\tau, \omega \in \Sigma_\sigma$ be such that $\tau/\sigma \preccurlyeq \omega/\sigma$. Then, by 1.3.7 a) there is an $\omega' \in \Sigma_\sigma$ with $\omega' \preccurlyeq \omega$ and $\tau/\sigma = \omega'/\sigma$, and injectivity of $\widehat{p}_\sigma$ yields $\tau = \omega' \preccurlyeq \omega$. Therefore, $\widehat{p}_\sigma^{-1}$ is increasing, and thus $\widehat{p}_\sigma$ is an isomorphism of ordered sets. □

(2.2.7) Corollary *Let Σ be a semifan in V, and let $\sigma \in \Sigma$.*

a) It holds

$$|\Sigma/\sigma| = p_\sigma\big(\bigcup \Sigma_\sigma\big), \quad \mathrm{cone}(\Sigma/\sigma) = p_\sigma\big(\mathrm{cone}\big(\bigcup \Sigma_\sigma\big)\big),$$

$$\langle \Sigma/\sigma \rangle = p_\sigma\big(\langle \bigcup \Sigma_\sigma \rangle\big), \quad \textit{and} \quad \dim(\Sigma/\sigma) = \dim\big(\langle \bigcup \Sigma_\sigma \rangle\big) - \dim(\sigma).$$

b) The isomorphism of ordered sets $\widehat{p}_\sigma : \Sigma_\sigma \to \Sigma/\sigma$ induces by restriction and coastriction bijections

$$\Sigma_\sigma \cap \Sigma_{k+\dim(\sigma)} \to (\Sigma/\sigma)_k \ \textit{ for every } k \in \mathbb{N}_0, \quad \Sigma_{\max} \cap \Sigma_\sigma \to (\Sigma/\sigma)_{\max},$$

$$\mathfrak{D}(\Sigma) \cap \Sigma_\sigma \to \mathfrak{D}(\Sigma/\sigma), \quad \textit{and } \mathfrak{F}(\Sigma) \cap \Sigma_\sigma \to \mathfrak{F}(\Sigma/\sigma).$$

c) If Σ is complete, equidimensional or equifulldimensional respectively, then so is Σ/σ.

d) If Σ/σ is full or fulldimensional respectively, then so is Σ.

Proof. a) and b) follow easily on use of 1.2.5 and 2.2.6. We show that if Σ is complete, then so is Σ/σ. So, let $x \in V \setminus 0$, and consider $[\![0, x]\!] \subseteq V = |\Sigma|$. Then, 1.2.29 implies the existence of $\tau \in \Sigma$ and $r \in]0, 1]$ such that $[\![0, rx]\!] \subseteq \tau$, and from this we get

$$p_\sigma(x) = \tfrac{1}{r} p_\sigma(rx) \in \tfrac{1}{r} p_\sigma(\tau) \subseteq |\Sigma/\sigma|.$$

Thus, Σ/σ is complete. The remaining claims follow immediately from a) and b). □

(2.2.8) Corollary *Let Σ be a nonempty semifan in V.*

a) It holds

$$|\Sigma/s(\Sigma)| = p_{s(\Sigma)}(|\Sigma|), \quad \mathrm{cone}(\Sigma/s(\Sigma)) = p_{s(\Sigma)}(\mathrm{cone}(\Sigma)),$$

$$\langle \Sigma/s(\Sigma) \rangle = p_{s(\Sigma)}(\langle \Sigma \rangle), \quad \textit{and} \quad \dim(\Sigma/s(\Sigma)) = \dim(\Sigma) - \dim(s(\Sigma)).$$

b) The isomorphism of ordered sets $\widehat{p}_{s(\Sigma)} : \Sigma \rightarrow \Sigma/s(\Sigma)$ induces by restriction and coastriction bijections

$$\Sigma_{k+\dim(s(\Sigma))} \rightarrow (\Sigma/s(\Sigma))_k \text{ for every } k \in \mathbb{N}_0, \quad \Sigma_{\max} \rightarrow (\Sigma/s(\Sigma))_{\max},$$

$$\mathfrak{D}(\Sigma) \rightarrow \mathfrak{D}(\Sigma/s(\Sigma)), \text{ and } \mathfrak{F}(\Sigma) \rightarrow \mathfrak{F}(\Sigma/s(\Sigma)).$$

c) Σ is complete, skeletal complete, full, fulldimensional, equidimensional, or equifulldimensional respectively if and only if $\Sigma/s(\Sigma)$ has the same property.

Proof. a) and b) are clear from 2.2.7. We show that if $\Sigma/s(\Sigma)$ is complete or skeletal complete respectively, then so is Σ. So, let $x \in V$. Then, there is a $\sigma \in \Sigma$ and a $y \in \sigma$, or a $y \in \operatorname{cone}(\Sigma)$, respectively, with $p_{s(\Sigma)}(x) = p_{s(\Sigma)}(y)$, and hence we have

$$x \in s(\Sigma) + y \subseteq s(\Sigma) + \sigma \subseteq \sigma \subseteq |\Sigma|$$

or

$$x \in s(\Sigma) + y \subseteq s(\Sigma) + \operatorname{cone}(\Sigma) \subseteq \operatorname{cone}(\Sigma),$$

respectively. Thus, Σ is complete or skeletal complete, respectively. The remaining claims follow immediately from a) and b). □

(2.2.9) Let Σ be a nonempty semifan in V, and let $\sigma \in \Sigma$. If Σ is skeletal complete, full, or fulldimensional respectively, then Σ/σ has not necessarily the same property. Moreover, if Σ/σ is complete, skeletal complete, equidimensional, or equifulldimensional respectively, then Σ has not necessarily the same property.

As counterexamples we consider the polycones $\sigma = \operatorname{cone}((1,0),(0,1))$, $\tau = \operatorname{cone}((-1,0),(0,1))$, $\rho = \operatorname{cone}((0,-1))$ and $\omega = \operatorname{cone}((-1,0))$ in $\mathbb{R}^2$. The fan with maximal cones σ, τ and ρ is skeletal complete and fulldimensional, but its projection along ρ is not full. The fan with maximal cones σ and τ is not skeletal complete, but its projection along $\sigma \cap \tau$ is complete. The fan with the maximal cones σ and ω is not equidimensional, but its projection along $\sigma \cap \tau$ is equifulldimensional.

(2.2.10) Let Σ be a nonempty W-semifan in V, and let $V' \subseteq V$ be a W-rational sub-$\mathbb{R}$-vector space such that $V' \cap \langle\Sigma\rangle = 0$. For every $\sigma \in \Sigma$, the sum $\sigma + V'$ is direct, and on use of 1.5.3 it is readily checked that $\Sigma' := \{\sigma + V' \mid \sigma \in \Sigma\}$ is a W-semifan in V with $s(\Sigma') = s(\Sigma) \oplus V'$.

(2.2.11) Proposition *Let Σ be a nonempty W-semifan in V. Then, there is a bijection from the set of W-extensions of Σ to the set of $W_{s(\Sigma)}$-extensions of $\Sigma/s(\Sigma)$ given by $\Sigma' \mapsto \widehat{p}_{s(\Sigma)}(\Sigma')$.*

Proof. If $\Sigma \subseteq \Sigma'$ is a W-extension, then it holds $s(\Sigma') = s(\Sigma)$ by 2.1.3 and hence $\widehat{p}_{s(\Sigma)}(\Sigma') = \Sigma'/s(\Sigma')$. Therefore, existence of the desired map follows from 2.2.6.

By 1.1.11 there exists a section of $p_{s(\Sigma)}$ in $\mathsf{Mod}(\mathbb{R})$ that is rational with respect to $W_{s(\Sigma)}$ and W. By means of this we consider $V_{s(\Sigma)}$ as a W-rational sub-$\mathbb{R}$-vector space of V, and then it holds $V_{s(\Sigma)} \cap s(\Sigma) = 0$. If $\Sigma/s(\Sigma) \subseteq \Sigma'$ is a $W_{s(\Sigma)}$-extension in $V_{s(\Sigma)}$, then we know from 2.2.10 that

$\alpha(\Sigma') := \{\sigma + s(\Sigma) \mid \sigma \in \Sigma/s(\Sigma)\}$ is a W-semifan in V with $s(\alpha(\Sigma')) = s(\Sigma)$, and it is clearly a W-extension of Σ. Now, it is easy to see that the map under consideration is bijective with inverse map given by $\Sigma' \mapsto \alpha(\Sigma')$, and thus the claim is proven. □

(2.2.12) Corollary *Let Σ be a nonempty W-semifan in V. Then, there is a bijection from the set of W-completions of Σ to the set of $W_{s(\Sigma)}$-completions of $\Sigma/s(\Sigma)$ given by $\Sigma' \mapsto \widehat{p}_{s(\Sigma)}(\Sigma')$.*

Proof. Clear by 2.2.11 and 2.2.8 c). □

(2.2.13) A semifan Σ in V is called *simplicial* if every cone in Σ is simplicial. Clearly, this is the case if and only if every $\sigma \in \Sigma_{\max}$ is simplicial, and then Σ is a fan.

A W-semifan Σ in V is called *W-regular* if every cone in Σ is W-regular, and then Σ is a fan. By 1.4.35 this is the case if and only if every $\sigma \in \Sigma_{\max}$ is W-regular. Moreover, W-regular W-fans are simplicial, and simplicial W-fans are W_K-regular. Hence, in case $R = K$, the properties of a W-fan to be W-regular, simplicial, or V-regular, are equivalent.

(2.2.14) Example Let $E \subseteq W$ be a basis of V, let $e_0 := -\sum_{e\in E} e$, and let

$$F := E \cup \{e_0\} \subseteq W.$$

It is readily checked that the W-polycone $\sigma_e := \operatorname{cone}(F \setminus \{e\})$ is W-regular for every $e \in F$. Hence, 1.4.27 implies $\sigma_e \cap \sigma_f = \operatorname{cone}(F \setminus \{e, f\}) \preccurlyeq \sigma_e$ for all $e, f \in F$, and therefore $\Omega := \bigcup_{e\in F} \operatorname{face}(\sigma_e)$ is a W-regular W-fan in V by 2.2.3.

Moreover, Ω is complete. Indeed, let $x \in V$. Then, there is a family $(r_e)_{e\in E}$ in $\mathbb{R}$ with $x = \sum_{e\in E} r_e e$. If $r_e \geq 0$ for every $e \in E$, then it holds $x \in \sigma_{e_0} \subseteq |\Omega|$. Otherwise, there exists an $f \in E$ such that $r_e - r_f \geq 0$ for every $e \in E$, and then we get

$$x = |r_f| e_0 + \textstyle\sum_{e\in E\setminus\{f\}} (r_e - r_f)e \in \sigma_f \subseteq |\Omega|.$$

Thus, Ω is complete.

2.3. Topological properties of fans

Let Σ be a semifan in V.

As mentioned above, the main result proven in this section will be a combinatorial description of the topological frontier of the support of a semifan. We start our topological investigations by introducing for a point x in the support of a semifan Σ the smallest cone in Σ containing the point x. Since the frontier of a full polycone consists of its proper faces by 1.4.21, this notion obviously has some topological significance.

(2.3.1) Let $x \in |\Sigma|$. Then, we denote by $\omega_{x,\Sigma} := \bigcap\{\sigma \in \Sigma \mid x \in \sigma\}$ the smallest cone in Σ containing x, and if no confusion can arise we write ω_x

instead of $\omega_{x,\Sigma}$. On use of 1.3.8 it is easily seen that for $\sigma \in \Sigma$ it holds $x \in \mathrm{in}_{\langle\sigma\rangle}(\sigma)$ if and only if $\sigma = \omega_x$, and in particular it holds $x \in \mathrm{in}_{\langle\omega_x\rangle}(\omega_x)$. Moreover, if $x \in \mathrm{fr}(|\Sigma|)$, then it is readily seen by 1.2.20 that $\dim(\omega_x) < n$.

(2.3.2) Proposition *Let $x \in \mathrm{fr}(|\Sigma|)$. Then, it holds $\omega_x \subseteq \mathrm{fr}(|\Sigma|)$.*

Proof. We start by showing that $\mathrm{in}_{\langle\omega_x\rangle}(\omega_x) \subseteq \mathrm{fr}(|\Sigma|)$. So, let $y \in \mathrm{in}_{\langle\omega_x\rangle}(\omega_x) \setminus \{x\}$, and let U be a neighbourhood of y in V. We have to show that $U \not\subseteq |\Sigma|$. Since $\bigcup\{\sigma \in \Sigma \mid y \notin \sigma\}$ is closed and does not contain y we can assume that every cone in Σ met by U contains y. Moreover, by 1.2.2 we can assume that U is convex.

Let L denote the affine line containing x and y. Then, there exists a $z \in L \cap \omega_x$ with $x \in \rrbracket z, y\rrbracket$. We set $C := \mathrm{conv}(\{z\} \cup U)$, and it is readily checked on use of convexity of U that $C = \bigcup_{q \in U} \llbracket z, q\rrbracket$. Hence, by 1.2.28 there exists a neighbourhood U' of x in V contained in C, and as $x \in \mathrm{fr}(|\Sigma|)$ there exists a $p \in U' \setminus |\Sigma|$. So, there is a $q \in U$ with $p \in \llbracket z, q\rrbracket$.

Now we assume that $U \subseteq |\Sigma|$. Then, there is a $\sigma \in \Sigma$ with $q \in \sigma$, and our assumption on U implies $y \in \sigma \cap \mathrm{in}_{\langle\omega_x\rangle}(\omega_x)$. As $\omega_x = \omega_y$ we get $z \in \omega_x \preccurlyeq \sigma$ and hence the contradiction $p \in \llbracket z, q\rrbracket \subseteq \sigma \subseteq |\Sigma|$.

So, it holds $\mathrm{in}_{\langle\omega_x\rangle}(\omega_x) \subseteq \mathrm{fr}(|\Sigma|)$, and the claim follows easily on use of 1.2.23 b). □

(2.3.3) Corollary *If Σ is nonempty, then it is not complete if and only if $0 \in \mathrm{fr}(|\Sigma|)$.*

Proof. Indeed, it is not complete if and only if $\mathrm{fr}(|\Sigma|)$ is nonempty, and therefore 2.3.2 implies the claim. □

In 2.2.7 c) we saw that projections of complete semifans are again complete, and we now set out to look for a converse of this result. We have to start with some preparations.

(2.3.4) Lemma *Let Σ be fulldimensional, and suppose that $\mathrm{fr}(|\Sigma|) = 0$. Then, it holds $n = 1$.*

Proof. Obviously it holds $n \geq 1$. Since $\mathrm{fr}(|\Sigma|)$ is nonempty, there is an $x \in V \setminus |\Sigma|$, and fulldimensionality of Σ implies by 1.2.20 the existence of a $y \in \mathrm{in}(|\Sigma|)$. Let U be a neighbourhood of y in V contained in $|\Sigma|$. If $z \in U$, then $\rrbracket z, x\llbracket$ meets $\mathrm{fr}(|\Sigma|)$ by 1.2.12 and hence contains 0, that is, we have $z \in \mathrm{cone}(-x)$. From this we get $U \subseteq \mathrm{cone}(-x)$ and therefore

$$1 \leq n = \dim(\langle U\rangle) \leq \dim(\mathrm{cone}(-x)) = 1$$

as desired. □

(2.3.5) Let $\sigma \in \Sigma$, and let $\Sigma' := \bigcup_{\tau \in \Sigma_\sigma} \mathrm{face}(\tau)$. Then, it holds $\Sigma'/\sigma = \Sigma/\sigma$. Moreover, for $\tau \in \Sigma_\sigma$ with $\tau \subseteq \mathrm{fr}(|\Sigma|)$ it is easily seen that $\tau \subseteq \mathrm{fr}(|\Sigma'|)$.

(2.3.6) Lemma *Let $x \in \mathrm{fr}(|\Sigma|)$, and let U be a neighbourhood of x in V. Moreover, let $A \subseteq V$ be closed and nowhere dense in V such that $x \in A$ and that for every $y \in A$ the affine sub-$\mathbb{R}$-space generated by $\{x, y\}$ is contained in A. Then, there exists a nonempty open subset $U' \subseteq U$ such that for every $y \in U' \setminus \{x\}$ it holds $\rrbracket x, y \rrbracket \cap (|\Sigma| \cup A) = \emptyset$.*

Proof. On use of 1.2.18 and 1.2.19 we can assume without loss of generality that $\bigcup \Sigma_{<n}$ is contained in A. Since $x \in \mathrm{fr}(|\Sigma|)$, there exists a $z \in U \setminus |\Sigma|$, and as $|\Sigma|$ is closed there exists a neighbourhood U'' of z in V contained in U and not meeting $|\Sigma|$. As A is nowhere dense in V, there is a $z' \in U'' \setminus A$, and as $|\Sigma| \cup A$ is closed there is a neighbourhood U' of z' in V contained in U'' and not meeting $|\Sigma| \cup A$.

Next, let $y \in U'$. First, we assume that there is a $y' \in \rrbracket x, y \rrbracket \cap A$. But then we have $x, y' \in A$, and this implies that y lies in the affine sub-$\mathbb{R}$-space generated by $\{x, y'\}$ and hence in A, contradicting that U' does not meet A. Next, we assume that $\rrbracket x, y \rrbracket$ meets $|\Sigma|$. Then, by the above and our assumptions on A there are $\sigma \in \Sigma_n$ and $y' \in \rrbracket x, y \rrbracket \cap \sigma$. But as $y \notin |\Sigma|$ and hence $y \notin \sigma$, it follows from 1.2.12 that there is a $y'' \in \rrbracket x, y \rrbracket \cap \mathrm{fr}(\sigma)$, and then 1.3.8 implies that there is a $\tau \in \sigma_{<n} \subseteq \Sigma_{<n}$ meeting $\rrbracket x, y \rrbracket$, contradicting what we have shown above. Thus, we get that $\rrbracket x, y \rrbracket \cap (|\Sigma| \cup A) = \emptyset$ for every $y \in U'$, and hence the claim is proven. □

(2.3.7) Analysing its proof, we see that 2.3.6 is still true if Σ is a finite set of closed conic subsets of V.

(2.3.8) Lemma *Let $r \in \mathbb{R}_{>0}$, and let $x, y, z, w \in V$ be such that $w \in \mathrm{conv}(x, y, z) \setminus (\llbracket x, y \rrbracket \cup \llbracket y, z \rrbracket)$. Then, $\rrbracket y + r(y - x), w \llbracket$ meets $\rrbracket y, z \llbracket$.*

Proof. Straightforward. □

(2.3.9) Lemma *Let V' be a further $\mathbb{R}$-vector space of finite dimension, let $f : V \to V'$ be a morphism in $\mathsf{Mod}(\mathbb{R})$, and let $\mathbb{A}$ be a set of subsets of V such that for every $A \in \mathbb{A}$ it holds $\mathrm{Ker}(f) \subseteq A$ and $A + A \subseteq A$. Then, it holds*

$$f\big(\mathrm{fr}\big(\bigcup \mathbb{A}\big)\big) \subseteq \mathrm{fr}\big(f\big(\bigcup \mathbb{A}\big)\big).$$

Proof. Setting $B := \bigcup \mathbb{A}$ and using continuity of f it is readily checked that

$$f(\mathrm{fr}(B)) = f(\mathrm{cl}(B) \cap \mathrm{cl}(V \setminus B)) \subseteq \mathrm{cl}(f(B)) \cap \mathrm{cl}(f(V \setminus B)) \subseteq$$

$$\mathrm{cl}(f(B)) \cap \mathrm{cl}(V' \setminus f(B)) = \mathrm{fr}(f(B)).$$ □

The main step in proving a converse of 2.2.7 c) as described above will be an answer to the following question: Consider a semifan Σ and cones σ and τ in Σ with $\tau \preccurlyeq \sigma$. If σ is contained in the frontier of $|\Sigma|$, is then σ/τ contained in the frontier of $|\Sigma/\tau|$? We will give a positive answer, in case

τ is 1-dimensional in 2.3.10, and in the general case by induction on the dimension of τ in 2.3.11.

(2.3.10) Lemma *Let $\sigma \in \Sigma$ be such that $\sigma \subseteq \mathrm{fr}(|\Sigma|)$, and let $\rho \in \sigma_1$. Then, it holds $\sigma/\rho \subseteq \mathrm{fr}(|\Sigma/\rho|)$.*

Proof. If $s(\Sigma) \neq 0$, then it holds $s(\Sigma) = \rho$, and the claim follows from 2.3.9. So, we can suppose that Σ is a fan. On use of 1.2.20 it is easy to see that $n > 1$. Moreover, by 2.3.5 we can replace Σ by $\bigcup_{\tau \in \Sigma_\rho} \mathrm{face}(\tau)$ and hence suppose that $\Sigma_{\max} \subseteq \Sigma_\rho$.

On use of 1.2.23 b) and continuity of p_ρ it is readily seen that it suffices to show that $p_\rho(\mathrm{in}_{\langle\sigma\rangle}(\sigma)) \subseteq \mathrm{fr}(|\Sigma/\rho|)$. So, let $x \in \mathrm{in}_{\langle\sigma\rangle}(\sigma)$. It follows from 2.3.1 that $\sigma = \omega_x$ and hence $\dim(\sigma) < n$. Let U be a neighbourhood of $p_\rho(x)$ in V_ρ. We have to show that U meets $V_\rho \setminus |\Sigma/\rho|$, and by 1.2.2 we can suppose U to be convex. Therefore, $p_\rho^{-1}(U)$ is a convex neighbourhood of x by 1.2.5. It follows from 2.3.6, 1.2.18, 1.2.19 and convexity of $p_\rho^{-1}(U)$ that there exists a $y \in p_\rho^{-1}(U) \setminus \{x\}$ such that $]\!]x, y]\!]$ is contained in $p_\rho^{-1}(U)$ and meets neither $|\Sigma|$ nor $\langle\sigma\rangle$.

Let $\tau \in \Sigma_{\max} \subseteq \Sigma_\rho$. Since σ is not full, the same holds for $\sigma \cap \tau$, and hence σ and τ are separable in their intersection by 1.4.14. Thus, there exists a linear hyperplane $H \subseteq V$ that separates σ and τ such that $\sigma \cap H = \sigma \cap \tau = \tau \cap H$. We denote by H_σ a closed linear halfspace defined by H containing σ, and by H_τ the other closed linear halfspace defined by H. Now, we show that there exists a $y_\tau \in]\!]x, y]\!]$ such that $\tau \cap (\rho + \mathrm{cone}(x, y_\tau)) \subseteq \sigma$. Indeed, if $\tau \cap (\rho + \mathrm{cone}(x, y)) \subseteq \sigma$, then $y_\tau := y$ fulfils the claim. So, suppose that $\tau \cap (\rho + \mathrm{cone}(x, y)) \nsubseteq \sigma$. We assume that $x \in \tau$ and hence $\sigma = \omega_x \preccurlyeq \tau$. Moreover, let $z \in \rho \setminus 0$ and let $w \in \tau \cap (\rho + \mathrm{cone}(x, y)) \setminus \sigma$. As $x \in \mathrm{in}_{\langle\sigma\rangle}(\sigma)$, there is an $r \in \mathbb{R}_{>0}$ such that $x + r(x - z) \in \mathrm{in}_{\langle\sigma\rangle}(\sigma) \subseteq \tau$, and as $w \in \rho + \mathrm{cone}(x, y) = \mathrm{cone}(z, x, y)$ it follows from 1.2.3 that there is an $s \in \mathbb{R}_{\geq 0}$ such that $sw \in \tau \cap \mathrm{conv}(z, x, y)$. But then, 2.3.8 yields the contradiction

$$\emptyset \neq]\!]x + r(x - z), sw[\![\cap]\!]x, y[\![\subseteq \tau \cap]\!]x, y]\!] \subseteq |\Sigma| \cap]\!]x, y]\!] = \emptyset.$$

Thus, it holds $x \in \sigma \setminus \tau$ and in particular $x \in H_\sigma \setminus H$. Furthermore, if $y \in H_\sigma$, then we get the contradiction

$$\tau \cap (\rho + \mathrm{cone}(x, y)) \subseteq \tau \cap H_\sigma = \tau \cap H = \tau \cap \sigma \subsetneq \sigma,$$

and hence it holds $y \in H_\tau \setminus H$. Now, by 1.2.12 there exists a $y_\tau \in]\!]x, y[\![\cap H$, and it follows

$$\tau \cap (\rho + \mathrm{cone}(x, y_\tau)) \subseteq \tau \cap H_\sigma = \tau \cap H = \tau \cap \sigma \subseteq \sigma$$

as desired.

The above being done for every $\tau \in \Sigma_{\max}$, it is easy to see that there exists a $y' \in]\!]x, y]\!]$ such that for every $\tau \in \Sigma_{\max}$ it holds

$$\tau \cap (\rho + \mathrm{cone}(x, y')) \subseteq \sigma.$$

Since $]\!]x, y]\!]$ does not meet $\langle\sigma\rangle$, it is readily checked that

$$(\rho + \mathrm{cone}(y')) \cap \sigma = \rho,$$

hence $\rho + y' \subseteq (\rho + \mathrm{cone}(x, y')) \setminus \sigma$, and therefore

$$(\rho + y') \cap \tau \subseteq \tau \cap (\rho + \mathrm{cone}(x, y')) \setminus \sigma = \emptyset$$

for every $\tau \in \Sigma_{\max}$. But from this we see that $\rho + y'$ does not meet $|\Sigma|$. Finally, assume that $\langle\rho\rangle + y'$ meets $|\Sigma|$. Then, there are $\tau \in \Sigma_{\max} \subseteq \Sigma_\rho$ and $z \in \rho$ with $y' - z \in \tau$, and this yields $y' = (y' - z) + z \in \tau + \rho \subseteq \tau$, hence $\rho + y' \subseteq \rho + \tau \subseteq \tau$ and therefore the contradiction that $y' + \rho$ is empty. Thus, $\langle\rho\rangle + y'$ does not meet $|\Sigma|$, and it is readily checked that $p_\rho(y') \in U$ is not contained in $|\Sigma/\rho|$. Herewith, the claim is proven. □

(2.3.11) Proposition *Let $\sigma \in \Sigma$, and let $\tau \preccurlyeq \sigma \subseteq \mathrm{fr}(|\Sigma|)$. Then, it holds $\sigma/\tau \subseteq \mathrm{fr}(|\Sigma/\tau|)$.*

Proof. On use of 2.2.6 and 2.3.9 we can assume without loss of generality that Σ is a fan, and then we prove the claim by induction on $d := \dim(\tau)$. If $d = 0$ it holds obviously, and if $d = 1$ it holds by 2.3.10. So, let $d > 1$, and suppose the claim to be true for strictly smaller values of d. By 1.4.17 there exists a $\rho \in \tau_1$ yielding $\tau/\rho \preccurlyeq \sigma/\rho \subseteq \mathrm{fr}(|\Sigma/\rho|)$ and hence

$$\sigma/\tau = (\sigma/\rho)/(\tau/\rho) \subseteq \mathrm{fr}(|(\Sigma/\rho)/(\tau/\rho)|) = \mathrm{fr}(|\Sigma/\tau|).$$

Thus, the claim is proven. □

Now we are ready to formulate and prove the desired result, characterising completeness of semifans in terms of their projections.

(2.3.12) Theorem *If $\dim(s(\Sigma)) \neq n - 1$ and $\Sigma_{\dim(s(\Sigma))+1} \neq \emptyset$, then the following statements are equivalent:*

(i) Σ is complete;

(ii) Σ/σ is complete for every $\sigma \in \Sigma_{\dim(s(\Sigma))+1}$.

Proof. By 2.2.8 c) we can assume without loss of generality that Σ is a fan and hence $n \neq 1$. The implication (i)⇒(ii) is clear by 2.2.7 c). So, suppose that Σ/σ is complete for every $\sigma \in \Sigma_1$. As $\Sigma_1 \neq \emptyset$ it follows from 2.2.7 c) that Σ is fulldimensional, and as $n \neq 1$ we get from 2.3.4 that $\mathrm{fr}(|\Sigma|) \neq 0$. Now, we assume that Σ is not complete, and hence there is an $x \in \mathrm{fr}(|\Sigma|) \setminus 0$. Then, 2.3.2 and 2.3.11 imply that Σ/ω_x is not complete. On the other hand, by 1.4.17 there is a $\rho \in (\omega_x)_1$, and on use of 2.2.7 c) we get the contradiction that $\Sigma/\omega_x = (\Sigma/\rho)/(\omega_x/\rho)$ is complete. Thus, the claim is proven. □

(2.3.13) It holds $\dim(s(\Sigma)) = n - 1$ if and only if Σ is of the form $\{u^\perp\}$, or $\{u^\perp, u^\vee\}$, or $\{u^\perp, u^\vee, (-u)^\vee\}$ for some $u \in V^* \setminus 0$, and it holds $\Sigma_{\dim(s(\Sigma))+1} = \emptyset$ if and only if Σ is empty or of the form $\{u^\perp\}$ for some $u \in V^*$. Using this it is easy to see that neither of these conditions can be dropped in 2.3.12.

Using the above characterisation of complete fans we are able to describe the frontier of an equifulldimensional semifan combinatorially. We begin with two auxiliary results.

(2.3.14) Lemma *Let Σ be equifulldimensional, and suppose that $\mathfrak{F}(\Sigma) = \emptyset$. Then, Σ is complete.*

Proof. By 2.2.8 b), c) we can assume without loss of generality that Σ is a fan, and then we prove the claim by induction on n. If $n \leq 1$, then it is clear. So, let $n > 1$, and suppose the claim to be true for strictly smaller values of n. From 2.2.7 b) it follows that Σ/σ is equifulldimensional and $\mathfrak{F}(\Sigma/\sigma) = \emptyset$ for every $\sigma \in \Sigma_1$, and hence Σ/σ is complete for every $\sigma \in \Sigma_1$. Then, 2.3.12 implies that Σ is complete, and thus the claim is proven. □

(2.3.15) Lemma *Let X be a topological space, and let $A, B \subseteq X$.*

a) Suppose that B is closed and that $\mathrm{in}(A) = \emptyset$. Then, it holds

$$\mathrm{in}(A \cup B) = \mathrm{in}(B).$$

b) Suppose that B is closed, that A is closed and nowhere dense in V, and that $A \cap \mathrm{in}(B) = \emptyset$. Then, it holds

$$\mathrm{fr}(A \cup B) = A \cup \mathrm{fr}(B).$$

Proof. a) Let $x \in \mathrm{in}(A \cup B)$. There is a neighbourhood U of x contained in $A \cup B$. As $U \setminus B$ is an open subset of A, it is empty, and hence we have $U \subseteq B$, yielding $x \in \mathrm{in}(B)$. The other inclusion being obvious, we get the claim.

b) By a) and 1.2.18 it holds

$$\mathrm{fr}(A \cup B) = (A \cup B) \setminus \mathrm{in}(B) = A \cup \mathrm{fr}(B). \qquad \square$$

(2.3.16) Proposition *The followings statements are equivalent:*

(i) Σ is equifulldimensional or empty;

(ii) It holds $\mathrm{fr}(|\Sigma|) = \bigcup \mathfrak{F}(\Sigma)$;

(iii) It holds $\mathrm{cl}(\mathrm{in}(|\Sigma|)) = |\Sigma|$.

Proof. Suppose that (i) holds. First, let $x \in \mathrm{fr}(|\Sigma|)$. Then, it holds $\omega_x \subseteq \mathrm{fr}(|\Sigma|)$ by 2.3.2, and hence Σ/ω_x is not complete by 2.3.11. It follows from 2.3.14 that $\mathfrak{F}(\Sigma/\omega_x)$ is nonempty. Now, 2.2.7 b) implies that $\mathfrak{F}(\Sigma) \cap \Sigma_{\omega_x}$ is nonempty, too, and thus there is a $\sigma \in \mathfrak{F}(\Sigma)$ such that $x \in \omega_x \subseteq \sigma \subseteq \bigcup \mathfrak{F}(\Sigma)$. So, it holds $\mathrm{fr}(|\Sigma|) \subseteq \bigcup \mathfrak{F}(\Sigma)$.

Conversely, let $\sigma \in \mathfrak{F}(\Sigma)$, and let $\tau \in \Sigma_n$ be such that $\sigma \preccurlyeq \tau$. We assume that $\sigma \not\subseteq \mathrm{fr}(|\Sigma|)$. Then, σ meets $\mathrm{in}(|\Sigma|)$, and by 1.2.24 there exists a $y \in \mathrm{in}_{\langle\sigma\rangle}(\sigma) \cap \mathrm{in}(|\Sigma|)$. On use of 1.2.24 it is easily seen that

$$y \notin \left(\bigcup \Sigma_n \setminus \{\tau\}\right) \cup \left(\bigcup \tau_{n-1} \setminus \{\sigma\}\right),$$

and therefore 1.1.6 and 1.2.21 yield the existence of a neighbourhood U of y in V, symmetric with respect to y and contained in $|\Sigma|$ such that

$U \cap \langle\sigma\rangle \subseteq \sigma$ and that τ and σ are the only cones in $\Sigma_{\geq n-1}$ met by U. As Σ is equifulldimensional this implies $U \subseteq \tau$, and on use of 1.4.21 it is easy to see that $U \setminus \langle\sigma\rangle \subseteq \mathrm{in}(\tau)$.

By 1.2.20 we get $U \not\subseteq \langle\sigma\rangle$, and hence there is a $z \in U \setminus \langle\sigma\rangle$. Furthermore, since U is symmetric with respect to y, there exists a $w \in U \setminus \langle\sigma\rangle$ with $y \in \rrbracket w, z \llbracket$. But now, 1.2.23 a) and 1.3.8 yield the contradiction $y \in \llbracket w, z \rrbracket \subseteq \mathrm{in}(\tau) \subseteq \tau \setminus \sigma$. Thus, (ii) is proven.

Now, suppose that (ii) holds. Since $|\Sigma|$ is closed, (iii) holds if $\bigcup \mathfrak{F}(\Sigma) \subseteq \mathrm{cl}(\mathrm{in}(|\Sigma|))$. So, let $\sigma \in \mathfrak{F}(\Sigma)$, let $x \in \sigma$, and let U be a neighbourhood of x in V. There is a $\tau \in \Sigma_n$ with $\sigma \prec \tau$ and hence $\sigma \subseteq \mathrm{fr}(\tau) = \mathrm{fr}(\mathrm{in}(\tau))$ by 1.3.8 and 1.2.23 b). Therefore, U meets $\mathrm{in}(\tau)$ and hence $\mathrm{in}(|\Sigma|)$, and thus we have $x \in \mathrm{cl}(\mathrm{in}(|\Sigma|))$. So, (iii) is proven.

Finally, suppose that (iii) holds. On use of 2.3.15 a) we get

$$\mathrm{in}(|\Sigma|) = \mathrm{in}\big(|\Sigma(n)| \cup \big(\bigcup \mathfrak{D}(\Sigma)\big)\big) = \mathrm{in}(|\Sigma(n)|),$$

and as (i) implies (ii) it follows $|\Sigma| = |\Sigma(n)|$. We assume that there is a $\sigma \in \mathfrak{D}(\Sigma)$. Then, it holds $\sigma = \bigcup_{\tau \in \Sigma_n} \sigma \cap \tau$, and moreover we have $\sigma \cap \tau \prec \tau$ for every $\tau \in \Sigma_n$. But by 1.3.8 this implies that σ is nowhere dense in $\langle\sigma\rangle$, contradictory to 1.2.20. So, we get $\mathfrak{D}(\Sigma) = \emptyset$ and hence (i). □

Finally, putting everything together we can prove the main result of this section, the desired combinatorial description of the frontier of a semifan.

(2.3.17) Theorem *It holds*

$$\mathrm{fr}(|\Sigma|) = \big(\bigcup \mathfrak{D}(\Sigma)\big) \cup \big(\bigcup \mathfrak{F}(\Sigma)\big).$$

Proof. By 2.1.9 and 2.1.10 it holds

$$|\Sigma| = \big(\bigcup \mathfrak{D}(\Sigma)\big) \cup (|\Sigma(n)|),$$

and we know from 1.2.21 that $\bigcup \mathfrak{D}(\Sigma)$ is closed and nowhere dense in V. Therefore, 2.1.10, 1.3.11 and 2.3.15 b) imply

$$\mathrm{fr}(|\Sigma|) = \big(\bigcup \mathfrak{D}(\Sigma)\big) \cup \mathrm{fr}(|\Sigma(n)|).$$

If Σ is not fulldimensional, then the claim follows from 2.1.9 and 2.1.10. If Σ is fulldimensional, then $\Sigma(n)$ is equifulldimensional by 2.1.10, and then the claim follows from 2.3.16. □

A first corollary is the converse of 2.3.11.

(2.3.18) Corollary *Let $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$. Then, it holds $\sigma \subseteq \mathrm{fr}(|\Sigma|)$ if and only if $\sigma/\tau \subseteq \mathrm{fr}(|\Sigma/\tau|)$.*

Proof. Clear from 2.3.17 and 2.2.7 b). □

(2.3.19) Corollary *It holds $\mathrm{fr}(|\Sigma/s(\Sigma)|) = p_{s(\Sigma)}(\mathrm{fr}(|\Sigma|))$.*

Proof. Clear from 2.3.18. □

(2.3.20) Corollary *Let $\sigma \in \Sigma$ be such that $\sigma \subseteq \mathrm{in}(|\Sigma|)$. Then, σ is the intersection of a family in Σ_n.*

Proof. There is a $\tau \in \Sigma_n$ with $\sigma \preccurlyeq \tau$, and hence σ is the intersection of a family in τ_{n-1} by 1.4.19. Therefore, we can assume without loss of generality that $\dim(\sigma) = n - 1$. Now, 2.3.17 implies that σ is contained in two different cones in Σ_n and hence equal to their intersection. □

We end this section with two further topological results, used in Section 3 for the construction of completions.

(2.3.21) Proposition *Let Ω be a complete semifan in V, let $X \subseteq V$ be closed, and let $Y := \mathrm{cl}(V \setminus X)$. Moreover, suppose that $\mathrm{cl}(\mathrm{in}(X)) = X$, and that every $\sigma \in \Omega$ is contained in X or in Y. Then, it holds*

$$X = \bigcup\{\sigma \in \Omega \mid \sigma \subseteq X\}, \quad Y = \bigcup\{\sigma \in \Omega \mid \sigma \subseteq Y\},$$

and

$$\mathrm{fr}(X) = \bigcup\{\sigma \in \Omega \mid \sigma \subseteq \mathrm{fr}(X)\}.$$

Proof. We set $\Omega_X := \{\sigma \in \Omega \mid \sigma \subseteq X\}$. Let $x \in \mathrm{in}(X)$. Completeness of Ω implies the existence of a $\sigma \in \Omega$ with $x \in \sigma$, and our hypothesis together with $\mathrm{in}(X) \cap Y = \emptyset$ yields $\sigma \in \Omega_X$. Therefore, we have $\mathrm{in}(X) \subseteq |\Omega_X|$ and hence $X = \mathrm{cl}(\mathrm{in}(X)) \subseteq |\Omega_X|$. The other inclusion being obvious, this shows the first equality, and the second equality holds for reasons of symmetry. The third equality now follows easily. □

(2.3.22) From 2.3.16 we see that 2.3.21 can be applied if X is the support of an equifulldimensional semifan in V.

(2.3.23) Proposition *Let $\Sigma \subseteq \Sigma'$ be an extension of semifans in V. Then, it holds $\sigma \cap \mathrm{in}(|\Sigma|) = \emptyset$ for every $\sigma \in \Sigma' \setminus \Sigma$.*

Proof. If σ meets $\mathrm{in}(|\Sigma|)$, then it follows from 1.2.24 that there are $\tau \in \Sigma$ and $x \in \mathrm{in}_{\langle\sigma\rangle}(\sigma) \cap \tau \subseteq \sigma \cap \tau$. But $\sigma \notin \Sigma$ implies $\sigma \cap \tau \prec \sigma$ and hence with 1.3.8 the contradiction $x \in \mathrm{fr}_{\langle\sigma\rangle}(\sigma)$. □

2.4. Subdivisions

Subdividing a fan Σ means replacing some cones in Σ (or better, facial fans of cones in Σ) by fans. Thereby one may or may not be allowed to add new 1-dimensional cones. The technique of subdivision can be used (and also will be used) to turn a nonsimplicial fan Σ into a simplicial one, by "splitting" the nonsimplicial cones of Σ into simplicial ones. We start by giving the definitions and make some first observations.

(2.4.1) Let Σ be a fan in V. A *W-subdivision of* Σ is a W-semifan Σ' in V with $|\Sigma| \subseteq |\Sigma'|$ such that for every $\sigma \in \Sigma'$ there is a $\tau \in \Sigma$ with $\sigma \subseteq \tau$, and a *strict W-subdivision of* Σ is a W-subdivision Σ' of Σ such that $\Sigma'_1 \subseteq \Sigma_1$. In case $W = V$ we speak just of (strict) subdivisions.

If Σ' is a W-subdivision of Σ then it is clear that it is a W-fan and that $|\Sigma| = |\Sigma'|$. Moreover, if Σ' is a strict subdivision of Σ, then it is a W-subdivision if and only if Σ is a W-fan.

If Σ' is a (strict) W-subdivision of Σ and Σ'' is a (strict) W-subdivision of Σ', then Σ'' is a (strict) W-subdivision of Σ.

(2.4.2) Proposition *Let Σ be a fan in V, and let Σ' be a subdivision of Σ. Then, it holds $\sigma = \bigcup\{\tau \in \Sigma' \mid \tau \subseteq \sigma\}$ for every $\sigma \in \Sigma$.*

Proof. Let $\sigma \in \Sigma$. For every $\tau \in \Sigma'$ it holds $\sigma \cap \tau \subseteq \sigma$, and there exists an $\omega \in \Sigma$ with $\tau \subseteq \omega$. Hence, it follows from 1.3.15 a) that $\sigma \cap \tau \preccurlyeq \tau$ and thus $\sigma \cap \tau \in \Sigma'$. Therefore, we get

$$\sigma = \sigma \cap |\Sigma'| = \bigcup\{\sigma \cap \tau \mid \tau \in \Sigma'\} = \bigcup\{\tau \in \Sigma' \mid \tau \subseteq \sigma\}. \qquad \square$$

(2.4.3) Corollary *Let Σ be a fan in V, and let Σ' be a subdivision of Σ. It holds $\Sigma_1 \subseteq \Sigma'_1$, and Σ' is a strict subdivision of Σ if and only if $\Sigma_1 = \Sigma'_1$.*

Proof. Let $\rho \in \Sigma_1$. By 2.4.2 it holds $\rho = \bigcup\{\tau \in \Sigma' \mid \tau \subseteq \rho\}$, and hence there is a $\tau \in \Sigma'$ with $\rho \subseteq \tau \subseteq \rho$, that is $\rho = \tau \in \Sigma'_1$. This proves the first claim, and the second is trivial. $\square$

Next, we show that subdivisions of a fan Σ induce in an obvious way subdivisions of all subfans of Σ.

(2.4.4) Proposition *Let Σ be a W-fan in V, let $\mathrm{T} \subseteq \Sigma$ be a subfan, let Σ' be a (strict) W-subdivision of Σ, and let*

$$\mathrm{T}' := \{\sigma \in \Sigma' \mid \exists \tau \in \mathrm{T} : \sigma \subseteq \tau\}.$$

Then, T' is a (strict) W-subdivision of T.

Proof. It is easy to see that T' is a W-fan in V and that every cone in T' is contained in a cone in T. For every $\sigma \in \mathrm{T}$ we clearly have

$$\sigma = \bigcup\{\tau \in \Sigma' \mid \tau \subseteq \sigma\} \subseteq |\mathrm{T}'|,$$

and hence it holds $|\mathrm{T}| \subseteq |\mathrm{T}'|$. Therefore, T' is a W-subdivision of T. If moreover Σ' is a strict W-subdivision of Σ and $\rho \in \mathrm{T}'_1$, then it holds $\rho \in \Sigma'_1 = \Sigma_1$ and hence $\rho \in \mathrm{T}_1$, proving the second claim. $\square$

(2.4.5) In the situation of 2.4.4, the W-subdivision T' of T is called *the W-subdivision of T induced by Σ'.* If Σ' is simplicial, then so is T'.

Now, we will prove the existence of simplicial strict subdivisions of arbitrary fans. We will do this by a recursive construction using the notion of direct sum of polycones from 1.5.

(2.4.6) Let Σ be a fan in V, and let $\rho \in \Sigma_1$. We set

$$\Sigma[\rho] := \{\rho + \tau \mid \exists \sigma \in \Sigma_\rho : \rho \not\preccurlyeq \tau \preccurlyeq \sigma\}$$

and $\Sigma(\rho) := (\Sigma \setminus \Sigma_\rho) \cup \Sigma[\rho]$. On use of 1.5.13 we get

$$\Sigma[\rho] = \{\rho \oplus \tau \mid \exists \sigma \in \Sigma_\rho : \rho \not\preccurlyeq \tau \preccurlyeq \sigma\},$$

and hence 1.5.4 implies $\Sigma(\rho) = (\Sigma \setminus \Sigma_\rho) \amalg \Sigma[\rho]$.

(2.4.7) Lemma *Let Σ be a W-fan in V, and let $\rho \in \Sigma_1$. Then, $\Sigma(\rho)$ is a strict W-subdivision of Σ.*

Proof. First we show that $\Sigma(\rho)$ is a W-semifan in V. By 1.4.6 it is clear that $\Sigma(\rho)$ is a finite set of W-polycones in V. Let $\omega \in \Sigma(\rho)$ and let $\omega' \preccurlyeq \omega$. If $\omega \in \Sigma \setminus \Sigma_\rho$, then it holds $\omega' \in \Sigma \setminus \Sigma_\rho \subseteq \Sigma(\rho)$ by 1.4.8. Otherwise, there are $\sigma \in \Sigma_\rho$ and $\tau \preccurlyeq \sigma$ with $\rho \not\preccurlyeq \tau$ such that $\omega = \rho \oplus \tau$, and 1.5.3 a) implies that there are $\rho' \preccurlyeq \rho$ and $\tau' \preccurlyeq \tau$ such that $\omega' = \rho' \oplus \tau'$. Then, we have either $\rho' = 0$ and hence $\omega' = \tau' \in \Sigma \setminus \Sigma_\rho \subseteq \Sigma(\rho)$, or $\rho' = \rho$ and hence $\omega' = \rho \oplus \tau' \in \Sigma[\rho] \subseteq \Sigma(\rho)$, as is easily seen on use of 1.4.8. Therefore, $\Sigma(\rho)$ is closed under taking faces.

Next, let $\omega, \omega' \in \Sigma(\rho)$. We have to show that $\omega \cap \omega' \preccurlyeq \omega$ and $\omega \cap \omega' \preccurlyeq \omega'$. If $\omega, \omega' \in \Sigma \setminus \Sigma_\rho$, then this is clear. So, consider the case that $\omega \in \Sigma \setminus \Sigma_\rho$ and that $\omega' \in \Sigma[\rho]$. Then, there are $\sigma \in \Sigma_\rho$ and $\tau \preccurlyeq \sigma$ with $\rho \not\preccurlyeq \tau$ such that $\omega' = \rho \oplus \tau$, and by 1.3.14 a) and 1.5.3 a) it follows

$$\omega \cap \omega' = \omega \cap \tau \in \operatorname{face}(\omega) \cap \operatorname{face}(\tau) \subseteq \operatorname{face}(\omega) \cap \operatorname{face}(\omega').$$

Now, consider the case that $\omega, \omega' \in \Sigma[\rho]$. Then, there are $\sigma, \sigma' \in \Sigma_\rho$, $\tau \preccurlyeq \sigma$ with $\rho \not\preccurlyeq \tau$, and $\tau' \preccurlyeq \sigma'$ with $\rho \not\preccurlyeq \tau'$ such that $\omega = \rho \oplus \tau$ and that $\omega' = \rho \oplus \tau'$. It follows from 1.3.14 b) and 1.5.3 a) that

$$\omega \cap \omega' = \rho \oplus (\tau \cap \tau') \in \operatorname{face}(\omega) \cap \operatorname{face}(\omega').$$

Thus, we have shown that $\Sigma(\rho)$ is a W-semifan in V.

It remains to show that $\Sigma(\rho)$ is a strict W-subdivision of Σ. By 1.4.23 we have

$$|\Sigma| = \left(\bigcup \Sigma \setminus \Sigma_\rho\right) \cup \left(\bigcup \{\rho + \tau \mid \exists \sigma \in \Sigma_\rho : \rho \not\preccurlyeq \tau \preccurlyeq \sigma\}\right) = |\Sigma(\rho)|,$$

and it is easy to see that every cone in $\Sigma(\rho)$ is contained in a cone in Σ. Finally, on use of 1.5.3 a) we get

$$\Sigma(\rho)_1 = (\Sigma_1 \setminus \{\rho\}) \cup \{\xi \in (\rho \oplus \tau)_1 \mid \exists \sigma \in \Sigma_\rho : \rho \not\preccurlyeq \tau \preccurlyeq \sigma\} =$$

$$\Sigma_1 \cup \{\xi \in \tau_1 \mid \exists \sigma \in \Sigma_\rho : \rho \not\preccurlyeq \tau \preccurlyeq \sigma\} = \Sigma_1,$$

and thus the claim is proven. □

(2.4.8) Theorem *If Σ is a W-fan in V, then there exists a simplicial strict W-subdivision of Σ.*

Proof. We choose a counting $(\rho_i)_{i=1}^r$ of Σ_1, and we set $\Sigma^{(0)} := \Sigma$ and $\Sigma^{(i)} := \Sigma^{(i-1)}(\rho_i)$ for every $i \in [1, r]$. Then, $\Sigma^{(r)}$ is a strict W-subdivision of Σ by 2.4.7 and 2.4.1. We assume that $\Sigma^{(r)}$ is not simplicial. From 1.5.3 a) and 1.5.8 it follows that there is an indecomposable $\sigma \in \Sigma^{(r)}$ with $\dim(\sigma) > 1$. Hence, the number

$$m := \max\{i \in [1, r] \mid \rho_i \preccurlyeq \sigma\}$$

exists, and we have $\sigma \in \Sigma^{(m-1)}[\rho_m]$. So, there is a $\tau \in \Sigma^{(m-1)}$ with $\sigma = \rho_m \oplus \tau$, and indecomposability of σ implies $\tau = 0$, hence $\sigma = \rho_m$ and thus the contradiction $\dim(\sigma) = 1$. Therefore, $\Sigma^{(r)}$ is simplicial and the claim is proven. □

(2.4.9) In the proof of 2.4.8 we choose a counting $(\rho_i)_{i=1}^{r}$ of Σ_1 to construct a simplicial strict W-subdivision $\Sigma^{(r)}$ of Σ. Considering the case that Σ is the facial fan of a nonsimplicial polycone in $\mathbb{R}^3$ it is readily checked that different countings of Σ_1 can lead to different subdivisions of Σ. So, we cannot make a uniqueness statement in 2.4.8.

3. Completions of fans

Let $R \subseteq \mathbb{R}$ be a subring, let K denote the field of fractions of R, let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, and let W be an R-structure on V.

The goal of this section is the Completion Theorem 3.7.5, stating that every (simplicial) semifan has a (simplicial) completion. Its proof is based on a sketch of Ewald and Ishida in [13]. To get a better understanding and also to provide a good basis for possible extensions of this result, we start by introducing a bunch of general notions and constructions, part of them being clearly of interest on their own. Putting them together at the end of the section will then finally yield the desired theorem. In order to motivate these notions and constructions we will give here a brief overview of the proof.

We start with a fan Σ (the generalisation to semifans is easy), and we look for a completion $\Sigma \subseteq \widehat{\Sigma}$. But we will prove more, namely the existence of a so-called *strong completion* of Σ, consisting of extensions $\Sigma \subseteq \overline{\Sigma} \subseteq \widehat{\Sigma}$ such that $\widehat{\Sigma}$ is complete and that $\overline{\Sigma}$ is a so-called *packing* of Σ. In order to explain and motivate the notions involved in this definitions, we start with the construction of $\widehat{\Sigma}$ out of $\overline{\Sigma}$.

Every $(n-1)$-dimensional cone in $\overline{\Sigma}$ generates a hyperplane, and every hyperplane arising in this way defines two closed halfspaces. The set of all intersections of these halfspaces is a complete semifan Ω, but not necessarily an extension of $\overline{\Sigma}$ (see 3.4). However, the cones in Ω contained in $\mathrm{cl}(V \setminus |\overline{\Sigma}|)$ form a semifan Ω' with support $\mathrm{cl}(V \setminus |\overline{\Sigma}|)$, and if $\overline{\Sigma}$ contains enough cones of dimension $n-1$ then Ω' is a fan. Now, Ω' contains a subfan T with support the frontier of $\overline{\Sigma}$, and this is a subdivision of the subfan $\overline{\mathfrak{F}}(\overline{\Sigma})$ (see 2.1.8) of $\overline{\Sigma}$. So, we may try to "adjust" $\overline{\Sigma}$ to T, aiming at an extension $\widetilde{\Sigma}$ of Σ with support $|\overline{\Sigma}|$ and $\overline{\mathfrak{F}}(\widetilde{\Sigma}) = \mathrm{T}$ (see 3.5). If we can do this, then $\widehat{\Sigma} := \widetilde{\Sigma} \cup \Omega'$ is a completion of Σ. But in order to do this, we need $\overline{\Sigma}$ to have certain additional properties. In particular, the cones in $\overline{\mathfrak{F}}(\overline{\Sigma})$ should be in some way "independent" from the cones in Σ in order to allow the above "adjustment". The key notions for this are *separability* and *tight separability* of extensions, introduced in 3.2, and *quasipackings*, introduced in 3.3: an extension $\Sigma \subseteq \Sigma'$ is separable if every cone in Σ' has a (necessarily unique) decomposition $\tau \oplus \tau'$ with $\tau \in \Sigma$ and such that τ' does meet Σ only in 0, and it is tightly separable if moreover the cones τ' in the above decompositions are contained in the topological frontier of Σ'. A quasipacking of Σ is an extension Σ' such that $|\Sigma|$ is contained in the topological interior of $|\Sigma'|$, with a possible exception of the origin. So, tightly separable quasipackings provide the above "independence" of the cones in $\overline{\mathfrak{F}}(\overline{\Sigma})$ from the cones in Σ. As we wish to get simplicial completions of simplicial fans, we will try to add only cones to Σ that are as simplicial as possible. In the above we can replace Ω by a simplicial subdivision, and moreover we can impose on $\overline{\Sigma}$ the condition that every cone in $\overline{\Sigma}$ that meets Σ only in 0, and in particular the components τ' in the decompositions coming from separability, is simplicial. Such an extension is called *relatively simplicial.* Altogether we define a packing to be a relatively simplicial, tightly separable quasipacking that fulfils an additional technical condition, and so we have sketched above how we can get a strong completion of Σ once we have a packing of Σ.

We will attack the problem of existence of strong completions by induction on the dimension n of the ambient space. The cases with $n \leq 1$ being obvious, it remains to prove the existence of packings in dimension n under the hypothesis of existence of strong completions in dimension $n-1$.

A general construction for doing so is given in 3.6 under the name of pullback. We choose a 1-dimensional cone ξ in Σ, and we consider the projection Σ/ξ of Σ along ξ. It may be instructive to think of $V/\langle\xi\rangle$ as embedded as an affine hyperplane in V, "orthogonal" to $\langle\xi\rangle$. By our induction hypothesis we find a strong completion $\Sigma/\xi \subseteq \overline{\mathrm{T}} \subseteq \widehat{\mathrm{T}}$ of Σ/ξ, and now we need to somehow "pull back" the cones in $\widehat{\mathrm{T}}$ and add them to Σ in a way that we end up with a relatively simplicial and tightly separable extension of Σ. The picture suggested above makes it clear that if we can do this well, then ξ will be contained in the interior of the new fan, and so we can construct a quasipacking recursively on the number of 1-dimensional cones lying in the frontier of Σ. Pulling back cones, or more generally fans, is done by choosing a so-called *pullback datum* (depending only on ξ) with an additional property, dubbed *very good* and depending on Σ and the extension of Σ/ξ that is wished to be pulled back (see 3.6.1 and 3.6.7). A key point in the section on pullbacks – and also for the whole proof – is the existence of very good pullback data, shown in 3.6.9 on use of a Hilbert norm (hence the picture suggested above).

It can be seen from examples in dimension 2 that fans cannot be completed canonically. Nevertheless, the construction described here tries to minimise the amount of choices, and moreover we try to keep the necessary choices visible throughout.

3.1. Relatively simplicial extensions

Let Σ be a fan in V.

We start by introducing the notion of polycones that are free over a fan Σ, auxiliary to the definition of relative simpliciality of extensions.

(3.1.1) Let σ be a polycone in V. By abuse of language we set

$$\sigma \cap \Sigma := \{\sigma \cap \tau \mid \tau \in \Sigma\}.$$

If $\sigma \cap \tau \preccurlyeq \tau$ for every $\tau \in \Sigma$, then $\sigma \cap \Sigma$ is a subfan of Σ.

A polycone σ in V is called *free over* Σ if $\sigma \cap \Sigma \subseteq \{0\}$. Clearly, a polycone $\sigma \in \Sigma$ is free over Σ if and only if $\sigma = 0$. A fan Σ' in V is called *free over* Σ if every $\sigma \in \Sigma'$ is free over Σ.

(3.1.2) Let Σ' be a fan in V such that for every $\tau' \in \Sigma'$ there is a $\tau \in \Sigma$ with $\tau' \subseteq \tau$, and let σ and σ' be polycones in V such that $\sigma' \subseteq \sigma$. If $\sigma \cap \tau = 0$ for every $\tau \in \Sigma_{\max}$, then σ' is free over Σ'.

From this it is seen that if σ is a polycone in V that is free over Σ, then every face of σ is free over every subfan of Σ. A further application shows that if Σ' is a fan in V that is free over Σ, then so is every subdivision of Σ'.

(3.1.3) Proposition *Let σ be a polycone in V with $\sigma \cap \tau \in \mathrm{face}(\sigma) \cap \mathrm{face}(\tau)$ for every $\tau \in \Sigma$. Then, the following statements are equivalent:*

(i) σ is free over Σ;

(ii) Every $\rho \in \sigma_1$ is free over Σ;

(iii) It holds $\sigma_1 \cap \Sigma = \emptyset$.

Proof. If σ is free over Σ then so is every $\rho \in \sigma_1$ by 3.1.2. Furthermore, if every $\rho \in \sigma_1$ is free over Σ and if moreover $\rho \in \sigma_1 \cap \Sigma$, then we get the contradiction that $\rho \in \sigma \cap \Sigma \subseteq \{0\}$. Finally, suppose that $\sigma_1 \cap \Sigma = \emptyset$ and that there is a $\tau \in \Sigma$ such that $\sigma \cap \tau \neq 0$. Then, there is a $\rho \in (\sigma \cap \tau)_1$, and since $\sigma \cap \tau \preccurlyeq \tau$ it holds $\rho \in \Sigma$ and therefore $\rho \notin \sigma_1$. But this implies the contradiction that $\sigma \cap \tau \not\preccurlyeq \sigma$. □

(3.1.4) If Σ' is a fan in V that is free over Σ, then $\Sigma \cup \Sigma'$ is an extension of Σ. Conversely, if $\Sigma \subseteq \Sigma'$ is an extension such that every polycone in $\Sigma' \setminus \Sigma$ is free over Σ, then $(\Sigma' \setminus \Sigma) \cup \{0\}$ is a subfan of Σ' that is free over Σ.

(3.1.5) Proposition *Let $\Sigma \subseteq \Sigma'$ be an extension, and let $\sigma \in \Sigma'$ be such that there is a finite family $(\sigma_i)_{i \in I}$ in Σ' such that $\sigma = \bigoplus_{i \in I} \sigma_i$. Then, σ_i is free over Σ for every $i \in I$ if and only if σ is so.*

Proof. If σ is not free over Σ, then by 3.1.3 and 1.5.3 a) we get the contradiction that there is a

$$\rho \in \sigma_1 \cap \Sigma = \bigcup_{i \in I} (\sigma_i)_1 \cap \Sigma = \emptyset.$$

The converse holds by 3.1.2 and 1.5.3 a). □

Now we can give the definition of relatively simplicial extensions. The idea is, as was mentioned at the beginning of this section, to extend the fan Σ by cones that are "as simplicial as possible".

(3.1.6) An extension Σ' of Σ is called *relatively simplicial (over Σ)* if every cone in Σ' that is free over Σ is simplicial. Since faces of simplicial polycones are again simplicial by 1.4.25, this is the case if and only if every maximal element of $\{\sigma \in \Sigma' \mid \sigma \cap \Sigma \subseteq \{0\}\}$ is simplicial. Clearly, Σ is relatively simplicial over itself, and every simplicial extension of Σ is relatively simplicial over Σ.

(3.1.7) A relatively simplicial extension of a simplicial fan is not necessarily simplicial. Indeed, if σ is a sharp polycone in $\mathbb{R}^3$ that is not simplicial and if

$\tau \preccurlyeq \sigma$ with $1 \leq \dim(\tau) \leq 2$, then the fan $\mathrm{face}(\tau)$ is simplicial and the extension $\mathrm{face}(\tau) \subseteq \mathrm{face}(\sigma)$ is relatively simplicial but not simplicial.

Additional conditions such that relatively simplicial extensions of simplicial fans are simplicial are given below in 3.2.7 and 3.3.6.

(3.1.8) Proposition *Let $\Sigma \subseteq \Sigma' \subseteq \Sigma''$ be extensions such that Σ'' is relatively simplicial over Σ. Then, Σ' is relatively simplicial over Σ, and Σ'' is relatively simplicial over Σ'.*

Proof. The first claim is obvious. The second claim holds since a polycone that is free over Σ' is free over every subfan of Σ' by 3.1.2. □

(3.1.9) Let $\Sigma \subseteq \Sigma' \subseteq \Sigma''$ be extensions such that the extensions $\Sigma \subseteq \Sigma'$ and $\Sigma' \subseteq \Sigma''$ are relatively simplicial. Then, Σ'' is not necessarily relatively simplicial over Σ. Indeed, if σ is a sharp polycone in $\mathbb{R}^3$ that is not simplicial and if $\tau \preccurlyeq \sigma$ with $1 \leq \dim(\tau) \leq 2$, then the extensions $\{0\} \subseteq \mathrm{face}(\tau) \subseteq \mathrm{face}(\sigma)$ provide a counterexample.

3.2. Separable extensions

Let Σ be a fan in V.

The idea of separability of an extension $\Sigma \subseteq \Sigma'$ is that the cones in Σ' are as independent of the cones in Σ as possible, as was discussed at the beginning of this section. We start by defining separability over Σ of a polycone.

(3.2.1) Let σ be a polycone in V, and let τ and τ' be polycones in V with $\sigma = \tau \oplus \tau'$. If[15] $\tau \in \Sigma \cup \{0\}$ and if moreover τ' is free over Σ, then it holds $\tau_1 = \sigma_1 \cap \Sigma$ and $\tau'_1 = \sigma_1 \setminus \Sigma$. Hence, on use of 1.4.17 we see that there is at most one pair (τ, τ') of polycones with $\sigma = \tau \oplus \tau'$ such that $\tau \in \Sigma \cup \{0\}$ and that τ' is free over Σ. If there is such a pair, then σ is called *separable over* Σ. Moreover, we set $\mathrm{in}_\Sigma(\sigma) := \tau$ and $\mathrm{ex}_\Sigma(\sigma) := \tau'$, and we call these polycones *the interior part of σ (with respect to Σ)* and *the exterior part of σ (with respect to Σ).*

Clearly, if σ is separable over Σ, then it is sharp, and it is a W-polycone if and only if its interior and exterior parts are W-polycones.

(3.2.2) Proposition *Let σ and τ be polycones in V.*

a) Suppose that $\tau \preccurlyeq \sigma$ and that σ is separable over Σ. Then, τ is separable over Σ, and it holds

$$\mathrm{in}_\Sigma(\tau) = \tau \cap \mathrm{in}_\Sigma(\sigma) \preccurlyeq \mathrm{in}_\Sigma(\sigma) \;\; \textit{and} \;\; \mathrm{ex}_\Sigma(\tau) = \tau \cap \mathrm{ex}_\Sigma(\sigma) \preccurlyeq \mathrm{ex}_\Sigma(\sigma).$$

b) Suppose that $\sigma \cap \tau \in \mathrm{face}(\sigma) \cap \mathrm{face}(\tau)$ and that σ and τ are separable over Σ. Then, $\sigma \cap \tau$ is separable over Σ, and it holds

$$\mathrm{in}_\Sigma(\sigma \cap \tau) = \mathrm{in}_\Sigma(\sigma) \cap \mathrm{in}_\Sigma(\tau) \;\; \textit{and} \;\; \mathrm{ex}_\Sigma(\sigma \cap \tau) = \mathrm{ex}_\Sigma(\sigma) \cap \mathrm{ex}_\Sigma(\tau).$$

[15] The condition on τ has to be of this form, rather than $\tau \in \Sigma$, to include the case that Σ is empty.

Proof. a) follows easily on use of 1.5.3 a), and b) follows from a). $\square$

Next we extend the above concept to extensions of fans.

(3.2.3) An extension Σ' of Σ is called *separable (over Σ)* if every cone in Σ' is separable over Σ. Then, we clearly have $\mathrm{ex}_\Sigma(\sigma) \in (\Sigma' \setminus \Sigma) \cup \{0\}$ for every $\sigma \in \Sigma'$. It follows from 3.2.2 a) that Σ' is separable over Σ if and only if every $\sigma \in \Sigma'_{\max}$ is separable over Σ. We set

$$\mathrm{ex}_\Sigma(\Sigma') := \{\mathrm{ex}_\Sigma(\sigma) \mid \sigma \in \Sigma'\}.$$

Obviously, Σ is separable over itself.

(3.2.4) Let $\Sigma \subseteq \Sigma'$ be an extension, and let $\sigma \in \Sigma'$ be such that there is a finite family $(\sigma_i)_{i\in I}$ in Σ' with $\sigma = \bigoplus_{i\in I} \sigma_i$. If σ_i is separable over Σ for every $i \in I$ and if moreover $\bigoplus_{i\in I} \mathrm{in}_\Sigma(\sigma_i) \in \Sigma$, then σ is separable over Σ, and it holds

$$\mathrm{in}_\Sigma(\sigma) = \bigoplus_{i\in I} \mathrm{in}_\Sigma(\sigma_i) \text{ and } \mathrm{ex}_\Sigma(\sigma) = \bigoplus_{i\in I} \mathrm{ex}_\Sigma(\sigma_i).$$

Indeed, this follows immediately from 3.1.5.

(3.2.5) Proposition *Let $\Sigma \subseteq \Sigma' \subseteq \Sigma''$ be extensions.*

a) If Σ'' is separable over Σ, then so is Σ'.

b) Suppose that Σ' is separable over Σ and that Σ'' is separable over Σ'. Then, Σ'' is separable over Σ, and for every $\sigma \in \Sigma''$ it holds

$$\mathrm{in}_\Sigma(\sigma) = \mathrm{in}_\Sigma(\mathrm{in}_{\Sigma'}(\sigma)) \text{ and } \mathrm{ex}_\Sigma(\sigma) = \mathrm{ex}_\Sigma(\mathrm{in}_{\Sigma'}(\sigma)) \oplus \mathrm{ex}_{\Sigma'}(\sigma).$$

Proof. a) holds obviously.

b) If $\Sigma = \Sigma'$ this is clear. So, we suppose that $\Sigma \neq \Sigma'$. Let $\sigma \in \Sigma''$. It holds $\mathrm{in}_{\Sigma'}(\sigma) \in \Sigma'$ and hence

$$\sigma = \mathrm{in}_\Sigma(\mathrm{in}_{\Sigma'}(\sigma)) \oplus \mathrm{ex}_\Sigma(\mathrm{in}_{\Sigma'}(\sigma)) \oplus \mathrm{ex}_{\Sigma'}(\sigma)$$

with $\mathrm{in}_\Sigma(\mathrm{in}_{\Sigma'}(\sigma)) \in \Sigma \cup \{0\}$. Moreover, $\mathrm{ex}_\Sigma(\mathrm{in}_{\Sigma'}(\sigma))$ is free over Σ, and so is $\mathrm{ex}_{\Sigma'}(\sigma)$ by 3.1.2. Then, 3.1.5 implies that $\mathrm{ex}_\Sigma(\mathrm{in}_{\Sigma'}(\sigma) \oplus \mathrm{ex}_{\Sigma'}(\sigma))$ is free over Σ, and the claim follows from this. $\square$

(3.2.6) Let $\Sigma \subseteq \Sigma' \subseteq \Sigma''$ be extensions such that Σ'' is separable over Σ. Then, Σ'' is not necessarily separable over Σ', as is seen by the example in 3.1.9.

The following result provides a first application of the notion of separability (see also 3.1.7).

(3.2.7) Proposition *If Σ is simplicial, then so is every relatively simplicial, separable extension of Σ.*

Proof. Clear by 1.5.8. $\square$

Finally we add a topological condition to get the notion of tight separability.

(3.2.8) A separable extension Σ' of Σ is called *tightly separable (over Σ)* if for every $\sigma \in \Sigma'$ it holds $\mathrm{ex}_\Sigma(\sigma) \setminus 0 \subseteq \mathrm{fr}_{\langle\Sigma'\rangle}(|\Sigma'|)$. Obviously, this is the case if and only if for every $\sigma \in \Sigma'_{\max} \setminus \Sigma$ it holds $\mathrm{ex}_\Sigma(\sigma) \setminus 0 \subseteq \mathrm{fr}_{\langle\Sigma'\rangle}(|\Sigma'|)$.

Clearly, Σ is tightly separable over itself.

(3.2.9) If Σ' is a tightly separable completion of Σ, then it holds $\Sigma' = \Sigma$. Indeed, since Σ' is complete it holds $\mathrm{ex}_\Sigma(\sigma) = 0$ for every $\sigma \in \Sigma'$ and hence $\Sigma' \subseteq \Sigma$.

(3.2.10) Proposition *Let $\Sigma \subseteq \Sigma' \subseteq \Sigma''$ be extensions such that Σ'' is tightly separable over Σ.*

a) If $\langle\Sigma'\rangle = \langle\Sigma''\rangle$, then Σ' is tightly separable over Σ.

b) If Σ'' is separable over Σ', then it is tightly separable over Σ'.

Proof. Claim a) holds since for every $\sigma \in \Sigma'$ we have

$$\mathrm{ex}_\Sigma(\sigma) \setminus 0 \subseteq |\Sigma'| \cap \mathrm{fr}_{\langle\Sigma''\rangle}(|\Sigma''|) \subseteq \mathrm{fr}_{\langle\Sigma'\rangle}(|\Sigma'|).$$

Claim b) holds since for every $\sigma \in \Sigma''$ it holds

$$\mathrm{ex}_{\Sigma'}(\sigma) \setminus 0 = \mathrm{ex}_{\Sigma'}(\mathrm{ex}_\Sigma(\sigma)) \setminus 0 \preccurlyeq \mathrm{ex}_\Sigma(\sigma) \setminus 0 \subseteq \mathrm{fr}_{\langle\Sigma''\rangle}(|\Sigma''|). \qquad \square$$

(3.2.11) Let $\Sigma \subseteq \Sigma' \subseteq \Sigma''$ be extensions such that Σ' is tightly separable over Σ and that Σ'' is tightly separable over Σ'. Then, Σ'' is not necessarily tightly separable over Σ. A counterexample is given by the fans $\Sigma'' = \mathrm{face}(\sigma) \cup \mathrm{face}(\tau)$, $\Sigma' = \mathrm{face}(\sigma)$ and $\Sigma = \mathrm{face}(\rho)$ in $\mathbb{R}^2$, where $\rho = \mathrm{cone}((1,0))$, $\sigma = \mathrm{cone}((1,0),(0,1))$ and $\tau = \mathrm{cone}((0,1),(-1,0))$.

3.3. Packings and strong completions

Let Σ be a fan in V.

The ideas of quasipackings and packings were described at the beginning of this section. We state now the precise definitions.

(3.3.1) Let $\Sigma \subseteq \Sigma'$ be an extension. Then, we set

$$C(\Sigma, \Sigma') := \{\rho \in \Sigma_1 \mid \Sigma'/\rho \text{ is not complete}\}$$

and $c(\Sigma, \Sigma') := \mathrm{Card}(C(\Sigma, \Sigma')) \in \mathbb{N}_0$. It is easily seen on use of 2.3.18 and 2.3.3 that $C(\Sigma, \Sigma') = \{\rho \in \Sigma_1 \mid \rho \subseteq \mathrm{fr}(|\Sigma'|)\}$.

(3.3.2) A W-extension Σ' of Σ is called a *W-quasipacking of Σ (in V)* if $|\Sigma| \setminus 0 \subseteq \mathrm{in}(|\Sigma'|)$. A W-quasipacking Σ' of Σ in V is called a *W-packing of Σ (in V)* if it is relatively simplicial and tightly separable over Σ and if moreover Σ'_1 is empty or Σ' is equifulldimensional in V. In case $W = V$ we speak just of quasipackings and packings of Σ.

(3.3.3) Example Since Σ is relatively simplicial and tightly separable over itself by 3.1.6 and 3.2.8, it is a packing of itself if and only if it is a quasipacking of itself. On use of 2.3.17 this is easily seen to be the case if and only if Σ is complete, if Σ_1 is empty, or if $n = 1$.

The following characterisation of quasipackings plays a crucial role in the construction of packings in 3.7.2: we will do this by descending recursion on $c(\Sigma, \Sigma')$, and the result below shows then that this yields what we wanted.

(3.3.4) Proposition *Let $\Sigma \subseteq \Sigma'$ be an extension. Then, the following statements are equivalent:*

(i) *Σ' is a quasipacking of Σ;*

(ii) *$c(\Sigma, \Sigma') = 0$.*

Proof. It is clear from 3.3.1 that (i) implies (ii). To show the converse we assume that $c(\Sigma, \Sigma') = 0$ and that $|\Sigma| \setminus 0 \not\subseteq \mathrm{in}(|\Sigma'|)$. Then, there is an $x \in |\Sigma| \setminus 0$ with $x \in \mathrm{fr}(|\Sigma'|)$, and by 2.3.2 it holds $0 \neq \omega_{x,\Sigma} = \omega_{x,\Sigma'} \subseteq \mathrm{fr}(|\Sigma'|)$. Then, 2.3.18 implies that Σ'/ρ is not complete for every $\rho \in (\omega_{x,\Sigma})_1$, and hence $(\omega_{x,\Sigma})_1 = \emptyset$, yielding the contradiction $\omega_{x,\Sigma} = 0$ on use of 1.4.17. Thus, the claim is proven. □

(3.3.5) Proposition *Let $\Sigma \subseteq \Sigma'$ be a quasipacking, and let $\Sigma' \subseteq \Sigma''$ be an extension. Then, every polycone in $\Sigma'' \setminus \Sigma'$ is free over Σ.*

Proof. For $\sigma \in \Sigma'' \setminus \Sigma'$ we have $\sigma \cap (|\Sigma| \setminus 0) \subseteq \sigma \cap \mathrm{in}(|\Sigma'|) = \emptyset$ by 2.3.23 and hence $\sigma \cap \Sigma \subseteq \{0\}$, and this shows the claim. □

(3.3.6) Corollary *Let $\Sigma \subseteq \Sigma'$ be a quasipacking, and let $\Sigma' \subseteq \Sigma''$ be an extension that is relatively simplicial over Σ. If Σ' is simplicial, then so is Σ''.*

Proof. Clear, since $\Sigma'' \setminus \Sigma'$ is free over Σ by 3.3.5. □

(3.3.7) Proposition *Let $\Sigma \subseteq \Sigma'$ be a quasipacking of Σ, and let $\sigma \in \Sigma'$.*

a) If $\sigma \subseteq \mathrm{fr}_{\langle\Sigma'\rangle}(|\Sigma'|)$, then σ is free over Σ.

b) If there exist $\tau \in \Sigma \cup \{0\}$ and $\tau' \in \Sigma'$ with $\tau' \subseteq \mathrm{fr}_{\langle\Sigma'\rangle}(|\Sigma'|)$ such that $\sigma = \tau \oplus \tau'$, then σ is separable over Σ with $\mathrm{in}_\Sigma(\sigma) = \tau$ and $\mathrm{ex}_\Sigma(\sigma) = \tau'$.

Proof. a) is obvious since $\mathrm{fr}_{\langle\Sigma'\rangle}(|\Sigma'|) \subseteq \mathrm{fr}(|\Sigma'|)$, and b) follows easily from a). □

(3.3.8) Proposition *a) If $\Sigma \subseteq \Sigma'$ is an equifulldimensional, tightly separable extension, then it holds $\mathrm{ex}_\Sigma(\Sigma') \subseteq \overline{\mathfrak{F}}(\Sigma')$.*

b) If $\Sigma \subseteq \Sigma'$ is a quasipacking, then it holds $\overline{\mathfrak{F}}(\Sigma') \subseteq \mathrm{ex}_\Sigma(\Sigma')$.

Proof. a) For $\sigma \in \Sigma'$ it holds $\mathrm{ex}_\Sigma(\sigma) \in \Sigma'$ and $\mathrm{ex}_\Sigma(\sigma) \setminus 0 \subseteq \mathrm{fr}(|\Sigma'|)$, and hence we have $\mathrm{ex}_\Sigma(\sigma) \in \overline{\mathfrak{F}}(\Sigma') \cup \{0\}$ by 2.3.16.

b) Let $\sigma \in \mathfrak{F}(\Sigma')$. Since $\mathrm{in}_\Sigma(\sigma) \subseteq |\Sigma| \cap \mathrm{fr}(|\Sigma'|) \subseteq \{0\}$ by 2.3.17 it follows $\mathrm{in}_\Sigma(\sigma) = 0$ and hence $\sigma = \mathrm{ex}_\Sigma(\sigma) \in \mathrm{ex}_\Sigma(\Sigma')$. Now, the claim follows by 3.2.2 a). □

(3.3.9) Corollary *If $\Sigma \subseteq \Sigma'$ is an equifulldimensional packing, then it holds $\mathrm{ex}_\Sigma(\Sigma') = \overline{\mathfrak{F}}(\Sigma')$.*

Proof. Clear from 3.3.8 □

At the end of this section we define strong completions, and we give some rather trivial examples. The last two of these will suit to start the inductive argument in constructing (strong) completions in 3.7.4.

(3.3.10) A *strong W-completion of Σ (in V)* is a pair $(\overline{\Sigma}, \widehat{\Sigma})$ consisting of a W-packing $\overline{\Sigma}$ of Σ in V and a W-completion $\widehat{\Sigma}$ of $\overline{\Sigma}$ in V that is relatively simplicial over Σ. In case $W = V$ we speak just of strong completions of Σ.

If $(\overline{\Sigma}, \widehat{\Sigma})$ is a strong completion of Σ, then it is clear from 3.3.5 and 3.2.5 b) that $\widehat{\Sigma}$ is separable over Σ.

(3.3.11) Example If $\overline{\Sigma}$ is a packing of Σ, then $(\overline{\Sigma}, \overline{\Sigma})$ is a strong completion of Σ if and only if $\overline{\Sigma}$ is complete. In particular, (Σ, Σ) is a strong completion of Σ if and only if Σ is complete, as follows on use of 3.3.3.

(3.3.12) Example If $\Sigma_1 = \emptyset$, and if Ω is a complete, simplicial W-fan in V, then (Σ, Ω) is a strong W-completion of Σ, as follows on use of 3.3.3.

(3.3.13) Example Let $n = 1$, let $x \in W \setminus 0$, let $\Sigma := \{0, \operatorname{cone}(x)\}$, and let $\widehat{\Sigma} := \{0, \operatorname{cone}(x), \operatorname{cone}(-x)\}$. Then, $(\Sigma, \widehat{\Sigma})$ is a strong W-completion of Σ. Indeed, this is easily seen on use of 3.3.3.

3.4. Construction of complete fans

We elaborate a general construction of complete semifans. A special case of this leads from a packing $\overline{\Sigma}$ of Σ to a fan Ω with support $\operatorname{cl}(V \setminus |\overline{\Sigma}|)$ and inducing a subdivision on $\overline{\mathfrak{F}}(\overline{\Sigma})$, as described at the beginning of this section.

(3.4.1) Let H be a finite set of W^*-rational lines in V^*. For $u = (u_L)_{L \in H} \in \prod_{L \in H}(L \setminus 0)$ it is clear that the finite set

$$\{(\bigcap_{L \in U} u_L^{\vee}) \cap (\bigcap_{L \in H \setminus U} -u_L^{\vee}) \mid U \subseteq H\}$$

of W-polycones in V depends not on the family u, but only on the set H; we denote it by Ω_H and we set $\overline{\Omega}_H := \bigcup_{\sigma \in \Omega_H} \operatorname{face}(\sigma)$.

Clearly, $\overline{\Omega}_H$ is a finite set of W-polycones in V that is closed under taking faces, and it is even a W-semifan in V. Indeed, by 2.2.3 it suffices to show that for all $\sigma, \tau \in \Omega_H$ it holds $\sigma \cap \tau \preccurlyeq \sigma$. So, let $u = (u_L)_{L \in H} \in \prod_{L \in H}(L \setminus 0)$, let $U, U' \subseteq H$, let

$$\sigma := (\textstyle\bigcap_{L \in U} u_L^{\vee}) \cap (\bigcap_{L \in H \setminus U} -u_L^{\vee}),$$

and let

$$\tau := (\textstyle\bigcap_{L \in U'} u_L^{\vee}) \cap (\bigcap_{L \in H \setminus U'} -u_L^{\vee}).$$

Then, it holds

$$\sigma \cap \tau = \sigma \cap (\textstyle\bigcap_{L \in U \setminus U'} u_L^{\perp}) \cap (\bigcap_{L \in U' \setminus U} u_L^{\perp}).$$

Let $(L_i)_{i=1}^r$ be a counting of $U \setminus U'$. For $k \in [0, r-1]$ it holds

$$\sigma \cap (\bigcap_{i=1}^k u_{L_i}^\perp) \subseteq \sigma \subseteq u_{L_{k+1}}^\vee$$

and

$$\sigma \cap (\bigcap_{i=1}^{k+1} u_{L_i}^\perp) = \sigma \cap (\bigcap_{i=1}^k u_{L_i}^\perp) \cap u_{L_{k+1}}^\perp,$$

and therefore we have $\sigma' := \sigma \cap (\bigcap_{L \in U \setminus U'} u_L^\perp) \preccurlyeq \sigma$. Now, let $(L'_i)_{i=1}^s$ be a counting of $U' \setminus U$. For $k \in [0, s-1]$ it holds

$$\sigma' \cap (\bigcap_{i=1}^k (-u_{L_i})^\perp) \subseteq \sigma' \subseteq \sigma \subseteq (-u_{L_{k+1}})^\vee$$

and

$$\sigma' \cap (\bigcap_{i=1}^{k+1} (-u_{L_i})^\perp) = \sigma' \cap (\bigcap_{i=1}^k (-u_{L_i})^\perp) \cap (-u_{L_{k+1}})^\perp,$$

and therefore we have $\sigma \cap \tau = \sigma' \cap (\bigcap_{L \in U' \setminus U} (-u_L)^\perp) \preccurlyeq \sigma'$, thus $\sigma \cap \tau \preccurlyeq \sigma$ as claimed.

Keeping the above notations, it is clear that for $x \in V$ and $L \in H$ it holds $x \in u_L^\vee$ or $x \in -u_L^\vee$, and setting $U := \{L \in H \mid x \in u_L^\vee\}$ we get $x \in (\bigcap_{L \in U} u_L^\vee) \cap (\bigcap_{L \in H \setminus U} -u_L^\vee) \in \overline{\Omega}_H$. This shows that the W-semifan $\overline{\Omega}_H$ is complete. We call $\overline{\Omega}_H$ *the complete W-semifan associated with H.*

(3.4.2) Proposition *Let H be a finite set of W^*-rational lines in V^*. The complete W-semifan $\overline{\Omega}_H$ associated with H is a fan if and only if $\bigcap_{L \in H} L^\perp = 0$.*

Proof. Using the notations from 3.4.1, we know from 2.3.20 that every $\sigma \in \overline{\Omega}_H \setminus \Omega_H$ is the intersection of a family in Ω_H and thus contains $\bigcap \Omega_H$. Therefore it holds $s(\overline{\Omega}_H) = \bigcap \Omega_H$, and since it is easily seen that $\bigcap \Omega_H = \bigcap_{L \in H} u_L^\perp$ the claim is proven. □

(3.4.3) Let Σ be a W-fan in V. Two polycones in $\mathfrak{F}(\Sigma)$ are W-separable in their intersection by 1.4.14. So, we can choose a family $H = (H_{\sigma,\tau})_{(\sigma,\tau) \in \mathfrak{F}(\Sigma)^2}$ of linear W-hyperplanes in V such that if $\sigma, \tau \in \mathfrak{F}(\Sigma)$, then $H_{\sigma,\tau}$ separates σ and τ in their intersection. Then, $\{H_{\sigma,\tau}^\perp \mid \sigma, \tau \in \mathfrak{F}(\Sigma)\}$ is a finite set of W^*-rational lines in V^*. The complete W-semifan associated with this set is called *the complete W-semifan associated with Σ and H* and is denoted by $\overline{\Omega}_{\Sigma,H}$. For $\sigma \in \mathfrak{F}(\Sigma)$ it holds $H_{\sigma,\sigma} = \langle \sigma \rangle$, and thus it follows from 3.4.2 that if $\bigcap_{\sigma \in \mathfrak{F}(\Sigma)} \langle \sigma \rangle = 0$, then $\overline{\Omega}_{\Sigma,H}$ is a fan.

We go on with some technical results needed for the application mentioned above.

(3.4.4) Lemma *Let Σ be a W-fan in V that is not complete. Then, there exists a simplicial, full W-polycone ω in V that is free over Σ.*

Proof. Since $V \setminus |\Sigma|$ is nonempty and open, it follows from 1.1.7, 1.1.8 and 1.2.2 that there exists a convex, open subset $U \subseteq V$ not meeting $|\Sigma|$ and containing a basis $E \subseteq W$ of V. Clearly, $\omega := \operatorname{cone}(E)$ is a full, simplicial W-polycone in V, and on use of 1.2.3 it is readily checked that ω is free over Σ. □

(3.4.5) Proposition *Let Σ be a W-fan in V that is not complete. Then, there exists a W-extension $\Sigma \subseteq \Sigma'$ with the following properties:*

i) Every cone in $\Sigma' \setminus \Sigma$ is simplicial;

ii) If Σ' is not complete, or if $n \neq 1$, then it holds $\mathfrak{F}(\Sigma) \subseteq \mathfrak{F}(\Sigma')$;

iii) If $H = (H_{\sigma,\tau})_{(\sigma,\tau)\in\mathfrak{F}(\Sigma')^2}$ is a family of linear W-hyperplanes in V such that if $\sigma, \tau \in \mathfrak{F}(\Sigma')$, then $H_{\sigma,\tau}$ separates σ and τ in their intersection, then $\overline{\Omega}_{\Sigma',H}$ is a W-fan.

Proof. By 3.4.4 there exists a simplicial, full W-polycone ω in V that is free over Σ, and then $\Sigma' := \Sigma \cup \mathrm{face}(\omega)$ is a W-extension of Σ by 3.1.2 and 3.1.4. Moreover, every cone in $\Sigma' \setminus \Sigma = \mathrm{face}(\omega) \setminus \{0\}$ is simplicial. It is easy to see that if Σ' is complete, then it holds $n \leq 1$, and that if $n \leq 1$, then the remaining claims are clear. So, suppose that $n > 1$.

Now, we show that $\mathfrak{F}(\Sigma) \subseteq \mathfrak{F}(\Sigma')$. If $\sigma \in \mathfrak{F}(\Sigma) \setminus \mathfrak{F}(\Sigma')$, then there are $\tau, \tau' \in \Sigma'_n \cap \Sigma'_\sigma$ with $\tau \neq \tau'$, hence $\tau \notin \Sigma$ or $\tau' \notin \Sigma$, and thus $\tau = \omega$ or $\tau' = \omega$. But this implies $\sigma \in \omega_{n-1} \cap \Sigma \subseteq \{0\}$ by 3.1.2 and therefore the contradiction $n = 1$. Hence, it holds $\mathfrak{F}(\Sigma) \subseteq \mathfrak{F}(\Sigma')$.

Next, let $\sigma \in \omega_{n-1}$, and let $\tau \in \Sigma'_n$ be such that $\sigma \preccurlyeq \tau$. If $\tau \neq \omega$, then it follows again $\sigma \in \omega_{n-1} \cap \Sigma \subseteq \{0\}$ by 3.1.2 and therefore the contradiction $n = 1$. Thus, it holds $\omega_{n-1} \subseteq \mathfrak{F}(\Sigma')$.

So, if E is a minimal W-generating set of ω, then by 1.4.27 we get $\bigcap_{\sigma\in\mathfrak{F}(\Sigma')}\langle\sigma\rangle \subseteq \bigcap_{\sigma\in\omega_{n-1}}\langle\sigma\rangle = \bigcap_{e\in E}\langle E \setminus \{e\}\rangle = 0$, and hence $\overline{\Omega}_{\Sigma',H}$ is a fan by 3.4.3 for every family H as described in iii). □

(3.4.6) Proposition *Let Σ be an equifulldimensional W-fan in V, let $\Sigma \subseteq \Sigma'$ be a W-extension such that $\mathfrak{F}(\Sigma) \subseteq \mathfrak{F}(\Sigma')$, and let $H = (H_{\sigma,\tau})_{(\sigma,\tau)\in\mathfrak{F}(\Sigma')^2}$ be a family of linear W-hyperplanes in V such that if $\sigma, \tau \in \mathfrak{F}(\Sigma')$, then $H_{\sigma,\tau}$ separates σ and τ in their intersection.*

a) For every $\sigma \in \overline{\Omega}_{\Sigma',H}$ it holds $\sigma \subseteq |\Sigma|$ or $\sigma \subseteq \mathrm{cl}(V \setminus |\Sigma|)$.

b) Let

$$\mathrm{T} := \{\sigma \in \overline{\Omega}_{\Sigma',H} \mid \sigma \subseteq \mathrm{cl}(V \setminus |\Sigma|)\} \text{ and } \mathrm{T}' := \{\sigma \in \overline{\Omega}_{\Sigma',H} \mid \sigma \subseteq \mathrm{fr}(|\Sigma|)\}.$$

Then, T is a W-semifan with $|\mathrm{T}| = \mathrm{cl}(V \setminus |\Sigma|)$ and $|\Sigma| \cup |\mathrm{T}| = V$, and T' is a subsemifan of T with $|\mathrm{T}'| = \mathrm{fr}(|\Sigma|)$.

c) T' is a W-subdivision of $\overline{\mathfrak{F}}(\Sigma)$.

Proof. a) Let $\sigma \in \overline{\Omega}_{\Sigma',H}$. We can assume without loss of generality that $\dim(\sigma) = n$. Suppose that $\sigma \nsubseteq \mathrm{cl}(V \setminus |\Sigma|)$, and assume that $\sigma \nsubseteq |\Sigma|$. Then, σ meets the open sets $\mathrm{in}(|\Sigma|)$ and $V \setminus |\Sigma|$, and therefore there exist $x \in \mathrm{in}(\sigma) \cap \mathrm{in}(|\Sigma|)$ and $y \in \mathrm{in}(\sigma) \setminus |\Sigma|$ by 1.2.24. Then, 1.2.12 yields the existence of a $z \in \rrbracket x, y \llbracket \cap \mathrm{fr}(|\Sigma|)$, and by 1.2.23 a) we have $z \in \mathrm{in}(\sigma)$. Since Σ is equifulldimensional, we know from 2.3.16 that there is a $\tau \in \mathfrak{F}(\Sigma) \subseteq \mathfrak{F}(\Sigma')$ with $z \in \tau$. But by construction of $\overline{\Omega}_{\Sigma',H}$ we know that σ lies on one side of $\langle\tau\rangle$, and hence $\mathrm{in}(\sigma)$ and in particular z lies strictly on one side of $\langle\tau\rangle$ by 1.1.14, contradicting $z \in \tau$. This proves the claim.

b) This follows easily by a), 2.3.21 and 2.3.22.

c) It is clear from b) that $|\mathrm{T}'| = |\overline{\mathfrak{F}}(\Sigma)|$, and we have to show that every cone in T' is contained in a cone in $\overline{\mathfrak{F}}(\Sigma)$. So, let $\tau \in \mathrm{T}' \subseteq \overline{\Omega}_{\Sigma',H}$. Then, we have $\tau \subseteq \bigcup \mathfrak{F}(\Sigma)$, and we assume that $\tau \not\subseteq \sigma$ for every $\sigma \in \mathfrak{F}(\Sigma)$, that is, there are $\sigma, \sigma' \in \mathfrak{F}(\Sigma)$ with $\sigma \neq \sigma'$ and $x \in \sigma \setminus \sigma'$ and $y \in \sigma' \setminus \sigma$ such that $x, y \in \tau$. Thus, $H_{\sigma,\sigma'}$ separates x and y strictly, too. But this contradicts τ lying on one side of $H_{\sigma,\sigma'}$, as holds since $\tau \in \overline{\Omega}_{\Sigma',H}$. Therefore, τ is contained in a cone in $\mathfrak{F}(\Sigma) \subseteq \overline{\mathfrak{F}}(\Sigma)$, and thus T' is a W-subdivision of $\overline{\mathfrak{F}}(\Sigma)$. □

3.5. Adjustments of extensions

Let $\Sigma \subseteq \Sigma'$ be a separable W-extension of fans in V, let $\mathrm{T} \subseteq \Sigma'$ be a subfan, and let T' be a W-subdivision of T.

In the overview of the proof at the beginning of this section we discussed adjusting fans to subdivisions of their frontiers. This procedure is examined in general now.

(3.5.1) For $\sigma \in \Sigma'$ and $\tau \in \mathrm{T}'$ it is clear that the sum

$$\mathrm{in}_\Sigma(\sigma) + (\mathrm{ex}_\Sigma(\sigma) \cap \tau)$$

is direct, and hence

$$\{\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) \mid \sigma \in \Sigma' \wedge \tau \in \mathrm{T}'\}$$

is a finite set of sharp W-polycones in V. We denote this set by $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ and call it *the adjustment of* Σ' *to* T' *over* Σ. On use of 1.3.15 a) it is easily seen that

$$\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}') = \{\mathrm{in}_\Sigma(\sigma) \oplus \tau \mid \sigma \in \Sigma' \wedge \tau \in \mathrm{T}' \wedge \tau \subseteq \mathrm{ex}_\Sigma(\sigma)\}.$$

The first question we address is if the set $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ is a fan.

(3.5.2) Proposition *a) For $\sigma \in \Sigma'$ and $\tau \in \mathrm{T}'$ it holds*

$$\mathrm{face}(\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau)) = \{\mathrm{in}_\Sigma(\sigma') \oplus (\mathrm{ex}_\Sigma(\sigma') \cap \tau') \mid \sigma' \preccurlyeq \sigma \wedge \tau' \preccurlyeq \tau\}.$$

b) For $\sigma, \sigma' \in \Sigma'$ and $\tau, \tau' \in \mathrm{T}'$ it holds

$$(\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau)) \cap (\mathrm{in}_\Sigma(\sigma') \oplus (\mathrm{ex}_\Sigma(\sigma') \cap \tau') =$$

$$\mathrm{in}_\Sigma(\sigma \cap \sigma') \oplus (\mathrm{ex}_\Sigma(\sigma \cap \sigma') \cap (\tau \cap \tau')).$$

Proof. a) Let $\omega \preccurlyeq \mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau)$. By 1.5.3 a) there are $\rho \preccurlyeq \mathrm{in}_\Sigma(\sigma)$ and $\tau' \preccurlyeq \mathrm{ex}_\Sigma(\sigma) \cap \tau$ such that $\omega = \rho \oplus \tau'$. Clearly, the sum $\sigma' := \rho + \mathrm{ex}_\Sigma(\sigma)$ is direct, and we have $\sigma' \preccurlyeq \sigma$ by 1.5.3 a). As $\mathrm{ex}_\Sigma(\sigma) \cap \tau \preccurlyeq \tau$ by 1.3.15 a) we have $\tau' \preccurlyeq \tau$. It is easily seen that $\mathrm{in}_\Sigma(\sigma') = \rho$ and $\mathrm{ex}_\Sigma(\sigma') = \mathrm{ex}_\Sigma(\sigma)$, and on use of this we get

$$\omega = \rho \oplus \tau' = \mathrm{in}_\Sigma(\sigma') \oplus (\mathrm{ex}_\Sigma(\sigma') \cap \tau')$$

as desired.

Conversely, let $\sigma' \preccurlyeq \sigma$ and $\tau' \preccurlyeq \tau$. By 3.2.2 a) we have $\mathrm{in}_\Sigma(\sigma') \preccurlyeq \mathrm{in}_\Sigma(\sigma)$ and $\mathrm{ex}_\Sigma(\sigma') \preccurlyeq \mathrm{ex}_\Sigma(\sigma)$, and by 1.3.15 a) we have $\mathrm{ex}_\Sigma(\sigma') \cap \tau' \preccurlyeq \tau'$ and $\mathrm{ex}_\Sigma(\sigma) \cap \tau \preccurlyeq \tau$. From this we get

$$\mathrm{ex}_\Sigma(\sigma') \cap \tau' = \mathrm{ex}_\Sigma(\sigma) \cap \tau \cap \mathrm{ex}_\Sigma(\sigma') \cap \tau' \preccurlyeq \mathrm{ex}_\Sigma(\sigma) \cap \tau,$$

and then 1.5.3 a) yields

$$\mathrm{in}_\Sigma(\sigma') \oplus (\mathrm{ex}_\Sigma(\sigma') \cap \tau') \preccurlyeq \mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau).$$

b) We set $\omega := \mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau)$, $\omega' := \mathrm{in}_\Sigma(\sigma') \oplus (\mathrm{ex}_\Sigma(\sigma') \cap \tau')$, and $\vartheta := \mathrm{in}_\Sigma(\sigma \cap \sigma') \oplus (\mathrm{ex}_\Sigma(\sigma \cap \sigma') \cap (\tau \cap \tau'))$. By 3.2.2 b) we see that

$$\vartheta = (\mathrm{in}_\Sigma(\sigma) \cap \mathrm{in}_\Sigma(\sigma')) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \mathrm{ex}_\Sigma(\sigma') \cap \tau \cap \tau') \subseteq$$

$$\omega \cap \omega' \subseteq \sigma \cap \sigma' = \mathrm{in}_\Sigma(\sigma \cap \sigma') \oplus \mathrm{ex}_\Sigma(\sigma \cap \sigma').$$

Now, let $x \in \omega \cap \omega'$. Then, there are $y \in \mathrm{in}_\Sigma(\sigma \cap \sigma')$, $y' \in \mathrm{ex}_\Sigma(\sigma \cap \sigma')$, $z \in \mathrm{in}_\Sigma(\sigma)$, $z' \in \mathrm{ex}_\Sigma(\sigma) \cap \tau$, $w \in \mathrm{in}_\Sigma(\sigma')$ and $w' \in \mathrm{ex}_\Sigma(\sigma') \cap \tau'$ such that $x = y + y' = z + z' = w + w'$. By 3.2.2 a) it follows that $y - z = z' - y' \in \langle \mathrm{in}_\Sigma(\sigma) \rangle \cap \langle \mathrm{ex}_\Sigma(\sigma) \rangle = 0$ and $y - w = w' - y' \in \langle \mathrm{in}_\Sigma(\sigma') \rangle \cap \langle \mathrm{ex}_\Sigma(\sigma') \rangle = 0$. Therefore, we get $y = z = w \in \mathrm{in}_\Sigma(\sigma \cap \sigma')$ and

$$y' = z' = w' \in \mathrm{ex}_\Sigma(\sigma \cap \sigma') \cap (\tau \cap \tau'),$$

hence $x \in \vartheta$. Herewith the claim is proven. □

(3.5.3) Corollary *The set* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is a* W*-fan in* V*.*

Proof. Clear from 3.5.1 and 3.5.2. □

Under mild hypotheses, in addition to being a fan the set $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ enjoys all the properties we looked for, as the next results will show.

(3.5.4) Proposition *a) If* $\mathrm{ex}_\Sigma(\Sigma') \subseteq \mathrm{T}$*, then* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is a* W*-subdivision of* Σ'*.*

b) If $\mathrm{ex}_\Sigma(\Sigma') \subseteq \mathrm{T}$ *and if moreover* T' *is a strict* W*-subdivision of* T*, then* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is a strict* W*-subdivision of* Σ'*.*

c) If $\mathrm{ex}_\Sigma(\Sigma') = \mathrm{T}$*, then the* W*-subdivisions of the subfans* $\Sigma \subseteq \Sigma'$ *and* $\mathrm{T} \subseteq \Sigma'$ *induced by* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *are* Σ *and* T'*, respectively.*

d) If $\mathrm{ex}_\Sigma(\Sigma') = \mathrm{T}$*, then* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is a* W*-extension of* Σ *and of* T'*.*

Proof. a) For $\sigma \in \Sigma'$ it holds

$$\sigma = \mathrm{in}_\Sigma(\sigma) \oplus \mathrm{ex}_\Sigma(\sigma) = \mathrm{in}_\Sigma(\sigma) + (\mathrm{ex}_\Sigma(\sigma) \cap |\mathrm{T}'|) =$$

$$\bigcup_{\tau \in \mathrm{T}'} (\mathrm{in}_\Sigma(\sigma) + (\mathrm{ex}_\Sigma(\sigma) \cap \tau)) \subseteq |\,\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')|,$$

and hence $|\Sigma'| \subseteq |\,\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')|$. If $\sigma \in \Sigma'$ and $\tau \in \mathrm{T}'$, then it holds

$$\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) \subseteq \mathrm{in}_\Sigma(\sigma) \oplus \mathrm{ex}_\Sigma(\sigma) = \sigma \in \Sigma',$$

and therefore every cone in $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ is contained in a cone in Σ'. This proves the claim.

b) Suppose that T' is a strict W-subdivision of T. Let $\rho \in \mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')_1$. Then, there are $\sigma \in \Sigma$ and $\tau \in \mathrm{T}'$ with $\tau \subseteq \mathrm{ex}_\Sigma(\sigma)$ such that $\rho = \mathrm{in}_\Sigma(\sigma) \oplus \tau$. By 1.5.1 we see that $\rho = \mathrm{in}_\Sigma(\sigma) \in \Sigma_1 \subseteq \Sigma'_1$ or $\rho = \tau \in \mathrm{T}'_1 = \mathrm{T}_1 \subseteq \Sigma'_1$, and thus the claim follows from a).

c) If T is empty, then so are T', Σ' and Σ, and then the claim is clear. So, suppose that T and hence T' is nonempty.

For $\tau \in \mathrm{T}'$ and $\sigma \in \Sigma$ we have $\sigma = \mathrm{in}_\Sigma(\sigma)$ and $\mathrm{ex}_\Sigma(\sigma) = 0$, hence $\sigma = \mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) \in \mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$, and this shows that $\Sigma \subseteq \mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$. Now, let $\sigma \in \Sigma'$ and $\tau \in \mathrm{T}'$, and let $\omega \in \Sigma$ with $\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) \subseteq \omega$. Every element of $\mathrm{ex}_\Sigma(\sigma) \cap \tau$ lies in $\omega \cap \tau = 0$, for T is free over Σ. This yields $\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) = \mathrm{in}_\Sigma(\sigma) \in \Sigma$, and therefore every cone in $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ contained in a cone in Σ is a cone in Σ. Thus, $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ induces Σ on Σ.

For $\tau \in \mathrm{T}'$ there is a $\sigma \in \Sigma'$ with $\tau \subseteq \mathrm{ex}_\Sigma(\sigma)$, and it holds $\mathrm{in}_\Sigma(\mathrm{ex}_\Sigma(\sigma)) = 0$ and $\mathrm{ex}_\Sigma(\mathrm{ex}_\Sigma(\sigma)) = \mathrm{ex}_\Sigma(\sigma)$, hence

$$\tau = \mathrm{ex}_\Sigma(\sigma) \cap \tau = \mathrm{in}_\Sigma(\mathrm{ex}_\Sigma(\sigma)) \oplus (\mathrm{ex}_\Sigma(\mathrm{ex}_\Sigma(\sigma)) \cap \tau) \in \mathrm{adj}_\Sigma(\Sigma', \mathrm{T}').$$

This shows that $\mathrm{T}' \subseteq \mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$. Now, let $\sigma \in \Sigma'$ and $\tau \in \mathrm{T}'$, and let $\omega \in \mathrm{T}$ with $\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) \subseteq \omega$. Every element of $\mathrm{in}_\Sigma(\sigma)$ lies in $\mathrm{in}_\Sigma(\sigma) \cap \omega = 0$, for T is free over Σ. This yields

$$\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) = \mathrm{ex}_\Sigma(\sigma) \cap \tau \preccurlyeq \tau$$

by 1.3.15 a) and hence $\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau) \in \mathrm{T}'$, and therefore every cone in $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ contained in a cone in T is a cone in T'. Thus, $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ induces T' on T.

d) is clear from c). □

(3.5.5) Corollary *Suppose that* $\mathrm{ex}_\Sigma(\Sigma') = \mathrm{T}$.

a) $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is separable over* Σ, *and for* $\sigma \in \Sigma'$ *and* $\tau \in \mathrm{T}'$ *it holds*

$$\mathrm{in}_\Sigma(\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau)) = \mathrm{in}_\Sigma(\sigma)$$

and

$$\mathrm{ex}_\Sigma(\mathrm{in}_\Sigma(\sigma) \oplus (\mathrm{ex}_\Sigma(\sigma) \cap \tau)) = \mathrm{ex}_\Sigma(\sigma) \cap \tau.$$

b) A cone in $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is free over* Σ *if and only if it is a cone in* T'.

c) $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is relatively simplicial over* Σ *if and only if* T' *is simplicial.*

Proof. This follows immediately from 3.5.1 and 1.3.15 a). □

(3.5.6) Corollary *Suppose that* $\mathrm{ex}_\Sigma(\Sigma') = \mathrm{T}$.

a) If Σ' *is tightly separable over* Σ, *then so is* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$.

b) If Σ' *is a quasipacking of* Σ, *then so is* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$.

c) If T' *is simplicial and* Σ' *is a packing of* Σ, *then* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *is a packing of* Σ.

Proof. a) is clear from 3.5.5 a), and b) is clear from 3.5.4. To show c), it suffices by 3.5.5 c) and by a) and b) to show that if $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')_1 \neq \emptyset$, then $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ is equifulldimensional. But if $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')_1 \neq \emptyset$, then we have $\Sigma'_1 \neq \emptyset$ by 3.5.4 a), and hence Σ' is equifulldimensional. Therefore, its subdivision $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ is equifulldimensional, too. $\square$

(3.5.7) Proposition *Suppose that* $\mathrm{ex}_\Sigma(\Sigma') = \mathrm{T}$, *and let* Ω *be a* W*-extension of* T' *such that* $|\Sigma'| \cap |\Omega| = |\mathrm{T}|$. *Then,* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}') \cup \Omega$ *is a* W*-extension of* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ *with* $|\,\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}') \cup \Omega| = |\Sigma'| \cup |\Omega|$, *and if* Ω *is simplicial, then* $\mathrm{adj}_\Sigma(\Sigma', \mathrm{T}') \cup \Omega$ *is relatively simplicial over* Σ.

Proof. We set $\widetilde{\Sigma} := \mathrm{adj}_\Sigma(\Sigma', \mathrm{T}')$ and $\widehat{\Sigma} := \widetilde{\Sigma} \cup \Omega$. It suffices to show that $\widehat{\Sigma}$ is a W-fan. It clearly is a finite set of W-polycones in V that is closed under taking faces, and since $\widetilde{\Sigma}$ and Ω both are fans it suffices to show that the intersection of a cone in $\widetilde{\Sigma}$ and a cone in Ω is a common face of both.

So, let $\sigma \in \Sigma'$, let $\tau \in \mathrm{T}'$ with $\tau \subseteq \mathrm{ex}_\Sigma(\sigma)$, and let $\omega \in \Omega$. Let $x \in (\mathrm{in}_\Sigma(\sigma) \oplus \tau)) \cap \omega$. Then, we have $x \in |\widetilde{\Sigma}| \cap |\Omega| = |\mathrm{T}| = |\mathrm{T}'|$ and hence $\omega_{x,\Omega} = \omega_{x,\mathrm{T}'} \in \mathrm{T}'$. Therefore, there is a $\vartheta \in \mathrm{T}$ with $\omega_{x,\Omega} \subseteq \vartheta$, and by hypothesis it holds $\mathrm{ex}_\Sigma(\vartheta) = \vartheta$. This shows that $\mathrm{in}_\Sigma(\vartheta) \oplus \omega_{x,\Omega}$ is a cone in $\widetilde{\Sigma}$, and then 3.5.2 b) implies

$$(\mathrm{in}_\Sigma(\sigma) \oplus \tau) \cap (\mathrm{in}_\Sigma(\vartheta) \oplus \omega_{x,\Omega}) = (\mathrm{in}_\Sigma(\sigma) \cap \mathrm{in}_\Sigma(\vartheta)) \oplus (\tau \cap \omega_{x,\Omega}).$$

On use of this and 3.2.2 it follows

$$x \in (\mathrm{in}_\Sigma(\sigma) \oplus \tau) \cap \omega_{x,\Omega} \subseteq (\mathrm{in}_\Sigma(\sigma) \oplus \tau) \cap (\mathrm{in}_\Sigma(\vartheta) \oplus \omega_{x,\Omega}) =$$

$$(\mathrm{in}_\Sigma(\sigma) \cap \mathrm{in}_\Sigma(\vartheta)) \oplus (\tau \cap \omega_{x,\Omega}) = \tau \cap \omega_{x,\Omega} \subseteq \tau.$$

Herewith we have shown that $(\mathrm{in}_\Sigma(\sigma) \oplus \tau) \cap \omega \subseteq \tau$ and hence

$$(\mathrm{in}_\Sigma(\sigma) \oplus \tau) \cap \omega = \tau \cap \omega.$$

Therefore, $(\mathrm{in}_\Sigma(\sigma) \oplus \tau) \cap \omega$ is a face of τ, hence of $\mathrm{in}_\Sigma(\sigma) \oplus \tau$ by 1.5.3 a), and of ω, and thus it is proven that $\widehat{\Sigma}$ is a W-extension of $\widetilde{\Sigma}$. The claim about the support of this extension is clear since $\widetilde{\Sigma}$ is a subdivision of Σ' by 3.5.4 a).

Finally, if Ω is simplicial, then so is T', and then $\widehat{\Sigma}$ is relatively simplicial over Σ by 3.5.5 c). $\square$

3.6. Pullbacks of extensions

Let Σ *be a* W*-fan in* V, *and let* $\xi \in \Sigma_1$. *We set* $p := p_\xi : V \twoheadrightarrow V_\xi$, $\widehat{p} := \widehat{p}_\xi$ *and* $\mathrm{T} := \Sigma/\xi$. *Moreover, we denote by* $\mathbb{L}$ *the set of* 1*-dimensional* W_ξ*-polycones in* V_ξ.

The pullback construction is perhaps the most technical one in the whole construction of completions. We begin by defining pullback data.

(3.6.1) A *W-pullback datum along ξ* is a triple (q, a, B) consisting of a section $q : V_\xi \rightarrowtail V$ of p in $\mathsf{Mod}(\mathbb{R})$ that is rational with respect to W_ξ and W, of a point $a \in W \cap \xi \setminus 0$, and of a family

$$B = (b_\rho)_{\rho \in \mathbb{L}} \in \prod_{\rho \in \mathbb{L}} (W_\xi \cap \rho \setminus 0).$$

Given a pullback datum along ξ, we will now define the pullback of a polycone in the quotient space modulo $\langle\xi\rangle$, supposed to be separable over T, and then make some first observations.

(3.6.2) Let (q, a, B) be a W-pullback datum along ξ, and let σ be a W_ξ-polycone in V_ξ that is separable over T. We set

$$B_\sigma := \{b_\rho \mid \rho \in \mathrm{ex}_{\mathrm{T}}(\sigma)_1\}$$

and

$$\psi_{q,a,B}(\sigma) := \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma)) + \mathrm{cone}(q(B_\sigma) + a).$$

It is clear that $\psi_{q,a,B}(\sigma)$ is a W-polycone in V, and if no confusion can arise we denote it just by $\psi(\sigma)$. It is readily checked that $p(\psi(\sigma)) = \sigma$. Moreover, if $\sigma = p(\tau)$ for some $\tau \in \Sigma_\xi$, then it clearly holds $\psi(\sigma) = \tau$.

(3.6.3) Lemma *Let (q, a, B) be a W-pullback datum along ξ, and let σ be a W_ξ-polycone in V_ξ that is separable over* T *such that* $\mathrm{ex}_{\mathrm{T}}(\sigma)$ *is simplicial.*

a) $B_\sigma \subseteq V_\xi$ is free, and $\mathrm{cone}(q(B_\sigma) + a)$ *is simplicial.*

b) The sums $\xi + \mathrm{cone}(q(B_\sigma) + a)$ and $\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma)) + \mathrm{cone}(q(B_\sigma) + a)$ are direct.

c) It holds $\xi \preccurlyeq \psi(\sigma)$, and $\psi(\sigma)$ is sharp.

d) It holds $\dim(\psi(\sigma)) = \dim(\sigma) + 1$.

Proof. a) follows easily from 1.4.25 and 1.4.15.

b) Since

$$p(\langle\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma))\rangle \cap \langle\mathrm{cone}(q(B_\sigma) + a)\rangle) \subseteq \langle\mathrm{in}_{\mathrm{T}}(\sigma)\rangle \cap \langle\mathrm{ex}_{\mathrm{T}}(\sigma)\rangle = 0$$

and hence

$$\langle\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma))\rangle \cap \langle\mathrm{cone}(q(B_\sigma) + a)\rangle \subseteq \langle\xi\rangle \cap \langle q(B_\sigma) + a\rangle$$

it suffices to prove the first claim. So, we assume that it is not true. Then we have $a \in \langle q(B_\sigma) + a\rangle$, and hence there is a family $(r_\rho)_{\rho\in\mathrm{ex}_{\mathrm{T}}(\sigma)_1}$ in $\mathbb{R}$ with $a = \sum_{\rho\in\mathrm{ex}_{\mathrm{T}}(\sigma)_1} r_\rho(q(b_\rho) + a)$. Applying p we get $\sum_{\rho\in\mathrm{ex}_{\mathrm{T}}(\sigma)_1} r_\rho b_\rho = 0$. As B_σ is free by a) it follows $b_\rho = 0$ for every $\rho \in \mathrm{ex}_{\mathrm{T}}(\sigma)_1$, and hence the contradiction $a = 0$.

c) follows from b) and 1.5.3 a).

d) On use of b), a), 2.2.7 b) and 1.5.1 we have

$$\dim(\psi(\sigma)) = \dim(\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma))) + \dim(\mathrm{cone}(q(B_\sigma) + a)) =$$

$$\dim(\mathrm{in}_{\mathrm{T}}(\sigma)) + 1 + \mathrm{Card}(B_\sigma) = \dim(\mathrm{in}_{\mathrm{T}}(\sigma)) + 1 + \dim(\mathrm{ex}_{\mathrm{T}}(\sigma)) = \dim(\sigma). \ \square$$

(3.6.4) Lemma *Let σ and τ be W_ξ-polycones in V_ξ that are separable over* T *such that* $\mathrm{ex}_{\mathrm{T}}(\sigma)$ *and* $\mathrm{ex}_{\mathrm{T}}(\tau)$ *are simplicial, and let (q, a, B) be a W-pullback datum along ξ.*

a) If $\sigma \cap \tau \in \mathrm{face}(\sigma) \cap \mathrm{face}(\tau)$, *then it holds*

$$\psi_{q,a,B}(\sigma) \cap \psi_{q,a,B}(\tau) = \psi_{q,a,B}(\sigma \cap \tau).$$

b) If $\tau \preccurlyeq \sigma$, then it holds

$$\psi_{q,a,B}(\sigma) \cap (\psi_{q,a,B}(\tau) - \xi) = \psi_{q,a,B}(\tau).$$

c) If $\tau \preccurlyeq \sigma$, then it holds $\psi_{q,a,B}(\tau) \preccurlyeq \psi_{q,a,B}(\sigma)$.

Proof. First we show the inclusion "$\supseteq$" in a). By 3.2.2 b) it holds $\mathrm{in}_{\mathrm{T}}(\sigma \cap \tau) = \mathrm{in}_{\mathrm{T}}(\sigma) \cap \mathrm{in}_{\mathrm{T}}(\tau)$ and $B_{\sigma\cap\tau} = B_\sigma \cap B_\tau$, and on use of 2.2.6 we therefore get

$$\psi(\sigma \cap \tau) = \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma)) \cap \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) + \mathrm{cone}(q(B_{\sigma\cap\tau}) + a) \subseteq \psi(\sigma) \cap \psi(\tau).$$

Next, we prove b). The inclusion "$\supseteq$" is clear by what we have shown above. So, let $x \in \psi(\sigma)$, let $y \in \psi(\tau)$, and let $z \in \xi$ be such that $x = y - z$. There are $x_0 \in \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma))$ and $x_1 \in \mathrm{cone}(q(B_\sigma) + a)$ with $x = x_0 + x_1$, and moreover there are

$$y_0 \in \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) \preccurlyeq \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma))$$

and

$$y_1 \in \mathrm{cone}(q(B_\tau) + a) \subseteq \mathrm{cone}(q(B_\sigma) + a)$$

with $y = y_0 + y_1$. Since $\xi \preccurlyeq \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma))$ we have $x_0 + z \in \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma))$, and as $(x_0 + z) + x_1 = y_0 + y_1$ it follows from 3.6.3 b) that $x_0 + z = y_0 \in \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau))$ and $x_1 = y_1 \in \mathrm{cone}(q(B_\tau) + a)$. Furthermore, there is a $u \in V^*$ such that $\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) = \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\sigma)) \cap u^\perp$, and since $\xi \preccurlyeq \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau))$ it holds $u(z) = 0$. This implies $u(x_0) = u(x_0 + z) = u(y_0) = 0$, hence $x_0 \in \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau))$ and thus $x \in \psi(\tau)$ as claimed.

Now, we prove the remaining inclusion in a). Let $x \in \psi(\sigma) \cap \psi(\tau)$. Then, by 3.6.2 we have

$$p(x) \in p(\psi(\sigma)) \cap p(\psi(\tau)) = \sigma \cap \tau = p(\psi(\sigma \cap \tau))$$

and thus $x \in \psi(\sigma) \cap (\psi(\sigma \cap \tau) - \xi)$. But now b) implies $x \in \psi(\sigma \cap \tau)$, and herewith a) is proven.

Finally, we prove c). There is a $u \in V_\xi^*$ with $\sigma \subseteq u^{\vee, V_\xi}$ such that $\tau = \sigma \cap u^{\perp, V_\xi}$. We set $v := u \circ p \in V^*$. Since $p(\psi(\sigma)) = \sigma$ by 3.6.2, we get $\psi(\sigma) \subseteq v^{\vee, V}$, and moreover

$$\psi(\sigma) \cap v^{\perp, V} = \{x \in \psi(\sigma) \mid p(x) \in p(\psi(\tau))\} = \psi(\sigma) \cap (\psi(\tau) - \xi) = \psi(\tau)$$

on use of b). This shows that $\psi(\tau) \preccurlyeq \psi(\sigma)$. □

Next, we start to pull back relatively simplicial, separable extensions of T, and we will show that the set of pulled back polycones form a fan.

(3.6.5) Let (q, a, b) be a W-pullback datum along ξ, and let T′ be a separable W_ξ-extension of T. We set

$$\Psi_{q,a,B}(\mathrm{T}') := \{\psi_{q,a,B}(\sigma) \mid \sigma \in \mathrm{T}'\}$$

and

$$\overline{\Psi}_{q,a,B}(\mathrm{T}') := \bigcup_{\sigma \in \Psi_{q,a,B}(\mathrm{T})} \mathrm{face}(\sigma),$$

and if no confusion can arise we denote these sets just by $\Psi(\mathrm{T}')$ and $\overline{\Psi}(\mathrm{T}')$, respectively. Furthermore, the set

$$\Sigma_{q,a,B}(\mathrm{T}') := \Sigma \cup \overline{\Psi}_{q,a,B}(\mathrm{T}')$$

is called *the pullback of* T' *over* Σ *along* ξ *by means of* (q, a, B), and if no confusion can arise it is denoted just by $\Sigma(\mathrm{T}')$.

(3.6.6) Proposition *Let* $\mathrm{T} \subseteq \mathrm{T}'$ *be a relatively simplicial, separable* W_ξ*-extension, and let* (q, a, B) *be a* W*-pullback datum along* ξ*. Then,* $\overline{\Psi}_{q,a,B}(\mathrm{T}')$ *is a* W*-fan in* V *such that* $\Sigma_\xi \subseteq \overline{\Psi}_{q,a,B}(\mathrm{T}')$ *and that* $\overline{\Psi}_{q,a,B}(\mathrm{T}')/\xi = \mathrm{T}'$.

Proof. We know from 3.6.2 that $\Psi(\mathrm{T}')$ is a finite set of W-polycones in V, and it follows from 3.6.4 that for all $\sigma, \tau \in \Psi(\mathrm{T}')$ it holds $\sigma \cap \tau \preccurlyeq \sigma$. Hence, $\overline{\Psi}(\mathrm{T}')$ is a W-semifan in V by 2.2.3. Moreover, it is easily seen that for every $\sigma \in \Sigma_\xi$ it holds $\sigma = \psi(\sigma/\xi) \in \Psi(\mathrm{T}')$, and then 2.2.1 implies in particular that $\overline{\Psi}(\mathrm{T}')$ is a fan. Finally, from 3.6.3 c) we get that for every $\sigma \in \mathrm{T}'$ it holds $\xi \preccurlyeq \psi(\sigma)$ and $\psi(\sigma)/\xi = p(\psi(\sigma)) = \sigma \in \mathrm{T}'$, and this implies $\overline{\Psi}(\mathrm{T}')/\xi = \mathrm{T}'$. □

The next question is, if the pulled back cones in $\Psi(\mathrm{T}')$ are compatible with the cones in Σ in the sense that their union $\Sigma(\mathrm{T}')$ is a fan. Some simple sketches show that for this to hold it suffices if the pulled back cones avoid the cones in Σ that do not contain ξ. In order to reach this we have to choose the pullback datum carefully, and for this we introduce the notion of good and very good pullback data. Right after we show that there are indeed very good pullback data.

(3.6.7) Let (q, a, B) be a W-pullback datum along ξ, and let $\mathrm{T} \subseteq \mathrm{T}'$ be a separable W_ξ-extension. Then, (q, a, B) is called *good for* Σ *and* T' if for every $\sigma \in \Sigma \setminus \overline{\Psi}_{q,a,B}(\mathrm{T}')$ and every $\tau \in \mathrm{T}'$ it holds

$$\sigma \cap \psi_{q,a,B}(\tau) = \sigma \cap \psi_{q,a,B}(\mathrm{in}_{\mathrm{T}}(\tau)),$$

and it is called *very good for* Σ *and* T' if it is good for Σ and T' and if moreover $\mathrm{cone}(q(B_\tau) + a)$ is free over Σ for every $\tau \in \mathrm{T}'$.

The following results 3.6.8 and 3.6.9 on the existence of very good pullback data are the only point in the proof of our Completion Theorem which makes use of a norm on V. One may compare this with the sketch of Ewald and Ishida mentioned at the beginning of this chapter.

(3.6.8) Lemma *Let* $\|\cdot\|$ *be a norm on* V*, let* $a \in \xi \setminus 0$*, and let* σ *and* τ *be polycones in* V *with* $\sigma \cap \tau \in \mathrm{face}(\sigma) \cap \mathrm{face}(\tau)$ *and* $\xi \in \mathrm{face}(\tau) \setminus \mathrm{face}(\sigma)$*. Then, there exists an* $\varepsilon_0 \in \mathbb{R}_{>0}$ *such that for every* $\varepsilon \in]0, \varepsilon_0[$ *and every finite subset* $B \subseteq V$ *with* $||b|| \leq \varepsilon$ *for every* $b \in B$ *it holds*

$$\sigma \cap (\tau + \mathrm{cone}(B + a)) = \sigma \cap \tau.$$

Proof. If $\sigma \cap \tau$ is full, then we get the contradiction $\xi \preccurlyeq \tau = \sigma \cap \tau = \sigma$. Therefore, $\sigma \cap \tau$ is not full, and hence σ and τ are separable in their intersection by 1.4.14. So, there exists a $u \in V^* \setminus 0$ such that $\sigma \subseteq u^\vee$, that $\tau \subseteq (-u)^\vee$, and that $\sigma \cap u^\perp = \sigma \cap \tau = \tau \cap u^\perp$. If $a \in u^\vee$, then we get $a \in \xi \cap u^\vee \subseteq \tau \cap u^\perp \subseteq \sigma$, therefore $\xi = \xi \cap \tau \cap \sigma \preccurlyeq \tau \cap \sigma \preccurlyeq \sigma$ and thus the contradiction $\xi \preccurlyeq \sigma$. Hence, it holds $a \notin u^\vee$, and denoting by d the distance on V induced by $\|\cdot\|$ we get $\varepsilon_0 := d(a, u^\vee) \in \mathbb{R}_{>0}$.

Now, let $\varepsilon \in]0, \varepsilon_0[$ and let $B \subseteq V$ be a finite subset with $||b|| \leq \varepsilon$ for every $b \in B$. If there is a $b \in B$ with $b+a \in u^\vee$, then we get the contradiction

$$\varepsilon \geq ||b|| = d(a, b+a) \geq d(a, u^\vee) = \varepsilon_0 > \varepsilon.$$

Therefore, it holds $B + a \subseteq (-u)^\vee$ and hence $\tau + \mathrm{cone}(B+a) \subseteq (-u)^\vee$. Now, this implies

$$\sigma \cap \tau \subseteq \sigma \cap (\tau + \mathrm{cone}(B+a)) \subseteq \sigma \cap u^\vee \cap (-u)^\vee = \sigma \cap u^\perp = \sigma \cap \tau$$

and thus the claim. □

(3.6.9) Proposition *Let* $\mathrm{T} \subseteq \mathrm{T}'$ *be a relatively simplicial, separable* W_ξ*-extension. Then, there exists a* W_K*-pullback datum along* ξ *that is very good for* Σ *and* T'*.*

Proof. Without loss of generality we can assume that $R = K$ and hence $W = W_K$. By 1.1.15 we can choose a W-rational Hilbert norm $\|\cdot\|$ on V, and we denote by d the distance induced by $\|\cdot\|$. Then, we know from 1.1.15 that $\|\cdot\|$ induces a Hilbert norm $\|\cdot\|'$ on V_ξ and that it moreover defines a section $q : V_\xi \rightarrowtail V$ of p in $\mathsf{Mod}(\mathbb{R})$ that is rational with respect to W_ξ and W and that induces by restriction and coastriction the canonical isomorphism of $\mathbb{R}$-Hilbert spaces from V_ξ onto the orthogonal complement of $\langle\xi\rangle$ in V with respect to $\|\cdot\|$.

Next, we can choose a point $a \in W \cap \xi \setminus 0$. For every $\sigma \in \Sigma \setminus \Sigma_\xi$ it holds $d(a, \sigma) > 0$, and finiteness of $\Sigma \setminus \Sigma_\xi$ implies the existence of an $\varepsilon_0 \in \mathbb{R}_{>0}$ such that for every $\sigma \in \Sigma \setminus \Sigma_\xi$ it holds $\varepsilon_0 \leq d(a, \sigma)$. Moreover, we know from 3.6.8 that there exists for every $\sigma \in \Sigma \setminus \Sigma_\xi$ and every $\tau \in \Sigma_\xi$ an $\varepsilon_{\sigma,\tau} \in \mathbb{R}_{>0}$ such that for every $\varepsilon \in]0, \varepsilon_{\sigma,\tau}[$ and for every finite subset $B \subseteq W_\xi$ such that $||b||' \leq \varepsilon$ for every $b \in B$, it holds $\sigma \cap (\tau + \mathrm{cone}(q(B) + a)) = \sigma \cap \tau$. Finiteness of $(\Sigma \setminus \Sigma_\xi) \times \Sigma_\xi$ implies the existence of an $\varepsilon_1 \in \mathbb{R}_{>0}$ such that for every $\sigma \in \Sigma \setminus \Sigma_\xi$ and every $\tau \in \Sigma_\xi$ it holds $\varepsilon_1 \leq \varepsilon_{\sigma,\tau}$. The hypothesis $R = K$ implies by 1.1.7 that W_ξ is dense in V_ξ, and hence there exists for every $\rho \in \mathbb{L}$ a $b_\rho \in W_\xi \cap \rho \setminus 0$ with $||b_\rho||' < \min\{\varepsilon_0, \varepsilon_1\}$. We set $B := (b_\rho)_{\rho \in \mathbb{L}}$, and then it is clear that (q, a, B) is a W-pullback datum along ξ.

Now, let $\sigma \in \Sigma \setminus \overline{\Psi}(\mathrm{T}')$, and let $\tau \in \mathrm{T}'$. As $\Sigma_\xi \subseteq \Psi(\mathrm{T}')$ it holds $\sigma \in \Sigma \setminus \Sigma_\xi$, and moreover we have $\psi(\mathrm{in}_\mathrm{T}(\tau)) = \widehat{p}^{-1}(\mathrm{in}_\mathrm{T}(\tau)) \in \Sigma_\xi$. Since B_τ is a finite subset of W_ξ such that for every $b \in B_\tau$ it holds $||b||' < \varepsilon_1 \leq \varepsilon_{\sigma, \widehat{p}^{-1}(\mathrm{in}_\mathrm{T}(\tau))}$, we thus have $\sigma \cap \psi(\tau) = \sigma \cap \psi(\mathrm{in}_\mathrm{T}(\tau))$, and therefore (q, a, B) is good for Σ and T'.

Finally, let $\sigma \in \Sigma$ and let $\tau \in \mathrm{T}'$. First, suppose that $\xi \preccurlyeq \sigma$ and hence $\sigma = \psi(p(\sigma))$ by 3.6.2. Since $\mathrm{ex}_\mathrm{T}(\tau)$ is free over T and as $p(\sigma) \in \mathrm{T}$, it holds $\mathrm{ex}_\mathrm{T}(\tau) \cap p(\sigma) = 0$ and hence $\psi(\mathrm{ex}_\mathrm{T}(\tau) \cap p(\sigma)) = \xi$. Therefore, as

$\mathrm{cone}(q(B_\tau)+a) \subseteq \psi(\mathrm{ex}_{\mathrm{T}}(\tau))$ and as $\mathrm{ex}_{\mathrm{T}}(\tau)$ is simplicial, it follows on use of 3.6.4 a) and 3.6.3 b) that

$$\mathrm{cone}(q(B_\tau)+a) \cap \sigma = \mathrm{cone}(q(B_\tau)+a) \cap \psi(\mathrm{ex}_{\mathrm{T}}(\tau)) \cap \psi(p(\sigma)) =$$

$$\mathrm{cone}(q(B_\tau)+a) \cap \psi(\mathrm{ex}_{\mathrm{T}}(\tau) \cap p(\sigma)) = \mathrm{cone}(q(B_\tau)+a) \cap \xi = 0.$$

Now, suppose that $\xi \not\preccurlyeq \sigma$, and let $x \in \mathrm{cone}(q(B_\tau)+a) \cap \sigma$. We assume that $x \neq 0$. By 1.2.3 there exists an $r \in \mathbb{R}_{>0}$ with $rx \in \mathrm{conv}(q(B_\tau)+a) \cap \sigma$, and hence there is a family $(r_\rho)_{\rho \in \mathrm{ex}_{\mathrm{T}}(\tau)_1}$ in $\mathbb{R}_{\geq 0}$ with $\sum_{\rho \in \mathrm{ex}_{\mathrm{T}}(\tau)_1} r_\rho = 1$ such that $rx = \sum_{\rho \in \mathrm{ex}_{\mathrm{T}}(\tau)_1} r_\rho(q(b_\rho)+a)$. But this implies the contradiction

$$d(a, rx) = ||rx - a|| = ||q(\textstyle\sum_{\rho \in \mathrm{ex}_{\mathrm{T}}(\tau)_1} r_\rho b_\rho)|| =$$

$$||\textstyle\sum_{\rho \in \mathrm{ex}_{\mathrm{T}}(\tau)_1} r_\rho b_\rho||' \leq \sum_{\rho \in \mathrm{ex}_{\mathrm{T}}(\tau)_1} r_\rho ||b_\rho||' < \varepsilon_0 \leq d(a, \sigma) \leq d(a, rx).$$

Thus, $\mathrm{cone}(q(B_\tau)+a)$ is free over σ, and hence (q, a, B) is very good for Σ and T'. $\square$

The next results state that good or very good pullback data are good or very good indeed, in the sense that they yield pullbacks with the desired properties.

(3.6.10) Proposition *Let $\mathrm{T} \subseteq \mathrm{T}'$ be a relatively simplicial, separable W_ξ-extension, and let (q, a, B) be a W-pullback datum along ξ.*

a) If (q, a, B) is good for Σ and T', then $\Sigma_{q,a,B}(\mathrm{T}')$ is a W-extension of Σ.

b) If (q, a, B) is very good for Σ and T', then $\Sigma_{q,a,B}(\mathrm{T}')$ is separable over Σ.

Proof. a) By 2.2.3 and 3.6.6 it suffices to show that for every $\sigma \in \Sigma \setminus \overline{\Psi}(\mathrm{T}')$ and every $\tau \in \mathrm{T}'$ it holds $\sigma \cap \psi(\tau) \in \mathrm{face}(\sigma) \cap \mathrm{face}(\psi(\tau))$. But as (q, a, B) is good for Σ and T' it holds $\sigma \cap \psi(\tau) = \sigma \cap \psi(\mathrm{in}_{\mathrm{T}}(\tau))$, and since $\psi(\mathrm{in}_{\mathrm{T}}(\tau)) \in \Sigma$ this implies

$$\sigma \cap \psi(\tau) \in \mathrm{face}(\sigma) \cap \mathrm{face}(\psi(\mathrm{in}_{\mathrm{T}}(\tau))).$$

Now, the claim follows on use of 3.2.2 a) and 3.6.4 c).

b) Let $\sigma \in \Sigma(\mathrm{T}')$. If $\sigma \in \Sigma$, then it is separable over Σ by 3.2.3. So, let $\sigma \notin \Sigma$. Then, there is a $\tau \in \mathrm{T}'$ with $\sigma \preccurlyeq \psi(\tau)$, and by 3.6.3 b) it holds $\psi(\tau) = \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) \oplus \mathrm{cone}(q(B_\tau)+a)$. Moreover, we have $\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) \in \Sigma$, and $\mathrm{cone}(q(B_\tau)+a)$ is free over Σ since (q, a, B) is very good for Σ and T'. Hence, $\psi(\tau)$ is separable over Σ, and thus so is σ by 3.2.5 b). $\square$

(3.6.11) Proposition *Let $\Sigma' \subseteq \Sigma$ be a subfan, let $\mathrm{T} \subseteq \mathrm{T}'$ be a relatively simplicial, separable W_ξ-extension, and let (q, a, B) be a W-pullback datum along ξ that is good for Σ and T'.*

a) $\Sigma_{q,a,B}(\mathrm{T}')$ is a W-extension of Σ'.

b) Suppose that (q, a, B) is very good for Σ and T'. If Σ is separable over Σ', then so is $\Sigma_{q,a,B}(\mathrm{T}')$.

c) If Σ is relatively simplicial over Σ', then so is $\Sigma_{q,a,B}(\mathrm{T}')$.

Proof. a) and b) are clear by 3.6.10 and 3.2.5 b). So, suppose that Σ is relatively simplicial over Σ', and let $\sigma \in \Sigma(\mathrm{T}')$ be free over Σ'. If $\sigma \in \Sigma$, then it is simplicial. So, suppose that $\sigma \notin \Sigma$ and hence

$$\sigma \preccurlyeq \psi(\tau) = \widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) \oplus \mathrm{cone}(q(B_\tau) + a)$$

for some $\tau \in \mathrm{T}'$ by 3.6.3 b). By 1.4.25 we can assume without loss of generality that $\sigma = \psi(\tau)$. Since σ is free over Σ, the same holds for $\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) \in \Sigma$ by 3.1.2, and therefore we have $\widehat{p}^{-1}(\mathrm{in}_{\mathrm{T}}(\tau)) = 0$ by 3.1.1. Then, 3.6.3 a) implies that $\sigma = \mathrm{cone}(q(B_\tau)+a)$ is simplicial, and thus the claim is proven. □

The only property of pullbacks lacking so far is tight separability. We prove this in 3.6.15, after some preparations.

(3.6.12) Proposition *Let $\mathrm{T} \subseteq \mathrm{T}'$ be a relatively simplicial, separable W_ξ-extension, and let (q, a, B) be a W-pullback datum along ξ that is good for Σ and T'. Moreover, we set $\overline{\Psi} := \overline{\Psi}_{q,a,B}(\mathrm{T}')$ and $\overline{\Sigma} := \Sigma_{q,a,B}(\mathrm{T}')$.*

a) It holds

$$\overline{\Psi}_{\max} = \{\psi(\sigma) \mid \sigma \in \mathrm{T}'_{\max}\}, \quad \overline{\Psi}_n = \{\psi(\sigma) \mid \sigma \in \mathrm{T}'_{n-1}\},$$

and

$$\mathfrak{D}(\overline{\Psi}) = \{\psi(\sigma) \mid \sigma \in \mathfrak{D}(\mathrm{T}')\}.$$

b) $\overline{\Psi}$ is equifulldimensional if and only if T' is so.

c) If T' is equifulldimensional, then it holds $\mathfrak{D}(\overline{\Sigma}) = \mathfrak{D}(\Sigma) \setminus \overline{\Psi}$ and

$$\mathfrak{F}(\overline{\Sigma}) = \{\sigma \in \mathfrak{F}(\Sigma) \mid \overline{\Psi}_\sigma \subseteq \Sigma_\sigma\} \cup \{\sigma \in \mathfrak{F}(\overline{\Psi}) \mid \Sigma_\sigma \subseteq \overline{\Psi}_\sigma\}.$$

Proof. a) follows easily on use of 3.6.4 and 3.6.3 d), b) is clear from a), and c) is clear from 2.1.11 and b). □

(3.6.13) Lemma *Let $\mathrm{T} \subseteq \mathrm{T}'$ be an equifulldimensional, relatively simplicial, separable W_ξ-extension, and let (q, a, B) be a W-pullback datum along ξ that is very good for Σ and T'. Let $\sigma \in \mathrm{T}' \setminus \mathrm{T}$ and let $\tau \in \Sigma_{q,a,B}(\mathrm{T}')$ be such that $\xi \not\preccurlyeq \tau$ and that*

$$\mathrm{cone}(q(B_\sigma) + a) \preccurlyeq \tau \preccurlyeq \psi_{q,a,B}(\sigma).$$

Then, it holds $\tau \subseteq \mathrm{fr}(|\Sigma_{q,a,B}(\mathrm{T}')|)$.

Proof. It holds $\tau \in \overline{\Psi}(\mathrm{T}') \setminus \overline{\Psi}(\mathrm{T}')_\xi$. Since T' is equifulldimensional, the same is true for $\overline{\Psi}(\mathrm{T}')$ by 3.6.12 b), and hence $\overline{\Psi}(\mathrm{T}')_\tau \cap \overline{\Psi}(\mathrm{T}')_n$ is not empty. Moreover, it holds $\overline{\Psi}(\mathrm{T}')_n \subseteq \overline{\Psi}(\mathrm{T}')_\xi$, and thus 2.1.12 b) yields the existence of a $\rho \in \mathfrak{F}(\overline{\Psi}(\mathrm{T}'))$ with $\tau \preccurlyeq \rho$. By 2.3.17 it suffices to show that $\rho \in \mathfrak{F}(\Sigma(\mathrm{T}'))$, and by 3.6.12 c) it suffices for this to show that $\Sigma_\rho \subseteq \overline{\Psi}(\mathrm{T}')_\rho$.

Let $\omega \in \Sigma(\mathrm{T}')_\tau$, and assume that $\omega \notin \overline{\Psi}(\mathrm{T}')$. Then, it holds $\omega \notin \Sigma_\xi$, and since (q, a, B) is good for Σ and T' we have $\omega \cap \psi(\sigma) = \omega \cap \psi(\mathrm{in}_{\mathrm{T}'}(\sigma))$. Together with $\tau \in \mathrm{face}(\psi(\sigma)) \cap \mathrm{face}(\omega)$ this implies

$$\mathrm{cone}(q(B_\sigma) + a) \preccurlyeq \tau \preccurlyeq \omega \cap \psi(\mathrm{in}_{\mathrm{T}}(\sigma)) \in \Sigma$$

and hence $\operatorname{cone}(q(B_\sigma)+a)=0$, since (q,a,B) is very good for Σ and T'. From this we get $\mathrm{ex}_{\mathrm{T}}(\sigma)=0$ and thus the contradiction $\sigma=\mathrm{in}_{\mathrm{T}}(\sigma)\in\mathrm{T}$. This shows that we have $\Sigma(\mathrm{T}')_\tau\subseteq\overline{\Psi}(\mathrm{T}')_\tau$ and thus

$$\Sigma_\rho\subseteq\Sigma(\mathrm{T}')_\rho\subseteq\Sigma(\mathrm{T}')_\tau\cap\Sigma(\mathrm{T}')_\rho\subseteq\overline{\Psi}(\mathrm{T}')_\tau\cap\Sigma(\mathrm{T}')_\rho=\overline{\Psi}(\mathrm{T}')_\rho$$

as desired. □

(3.6.14) Corollary *Let $\Sigma'\subseteq\Sigma$ be a subfan such that Σ is separable over Σ', and let $\mathrm{T}\subseteq\mathrm{T}'$ be an equifulldimensional, relatively simplicial, separable W_ξ-extension. Moreover, let (q,a,B) be a W-pullback datum along ξ that is very good for Σ and T'. Let $\sigma\in\mathrm{T}'\setminus\mathrm{T}$ and let $\tau\in\Sigma_{q,a,B}(\mathrm{T}')$ be such that $\mathrm{ex}_{\Sigma'}(\tau)\preccurlyeq\psi_{q,a,B}(\sigma)$. Then, it holds $\mathrm{ex}_{\Sigma'}(\tau)\subseteq\mathrm{fr}(|\Sigma_{q,a,B}(\mathrm{T}')|)$.*

Proof. Since $\mathrm{ex}_{\Sigma'}(\tau)$ is free over Σ', it holds $\mathrm{ex}_{\Sigma'}(\tau)\preccurlyeq\mathrm{ex}_{\Sigma'}(\psi(\sigma))$, and hence it suffices to show that $\mathrm{ex}_{\Sigma'}(\psi(\sigma))\subseteq\mathrm{fr}(|\Sigma(\mathrm{T}')|)$. But since $\operatorname{cone}(q(B_\sigma)+a)\preccurlyeq\mathrm{ex}_{\Sigma'}(\psi(\sigma))\preccurlyeq\psi(\sigma)$ and $\xi\not\preccurlyeq\mathrm{ex}_{\Sigma'}(\psi(\sigma))$ this follows from 3.6.13. □

(3.6.15) Proposition *Let $\Sigma'\subseteq\Sigma$ be a subfan, and let $\mathrm{T}\subseteq\mathrm{T}'$ be an equifulldimensional, relatively simplicial, separable W_ξ-extension. Moreover, let (q,a,B) be a W-pullback datum along ξ that is very good for Σ and T'. If Σ is tightly separable over Σ', then so is $\Sigma_{q,a,B}(\mathrm{T}')$.*

Proof. Since (q,a,B) is very good for Σ and T', we know from 3.6.11 b) that $\Sigma(\mathrm{T}')$ is separable over Σ'. Let $\sigma\in\Sigma(\mathrm{T}')$. We have to show that $\mathrm{ex}_{\Sigma'}(\sigma)\setminus 0\subseteq\mathrm{fr}(|\Sigma(\mathrm{T}')|)$. If $\sigma\notin\Sigma$, then there is a $\tau\in\mathrm{T}'\setminus\mathrm{T}$ with $\sigma\preccurlyeq\psi(\tau)$, and then the claim follows by 3.6.14. So, suppose that $\sigma\in\Sigma$.

If there exists an $\omega\in\mathfrak{F}(\Sigma)$ with $\overline{\Psi}(\mathrm{T}')_\omega\subseteq\Sigma_\omega$ and $\mathrm{ex}_{\Sigma'}(\sigma)\preccurlyeq\omega$, then it holds $\omega\in\mathfrak{F}(\Sigma(\mathrm{T}'))$ by 3.6.12 c) and hence $\mathrm{ex}_{\Sigma'}(\sigma)\subseteq\omega\subseteq\mathrm{fr}(|\Sigma(\mathrm{T}')|)$ by 2.3.17. If there exists an $\omega\in\mathfrak{D}(\Sigma)\setminus\overline{\Psi}(\mathrm{T}')$ with $\mathrm{ex}_{\Sigma'}(\sigma)\preccurlyeq\omega$, then it holds $\omega\in\mathfrak{D}(\Sigma(\mathrm{T}'))$ by 3.6.12 c) and hence $\mathrm{ex}_{\Sigma'}(\sigma)\subseteq\omega\subseteq\mathrm{fr}(|\Sigma(\mathrm{T}')|)$ by 2.3.17.

So, suppose that $\mathrm{ex}_{\Sigma'}(\sigma)$ is not a face of some $\omega\in\mathfrak{F}(\Sigma)$ with $\overline{\Psi}(\mathrm{T}')_\omega\subseteq\Sigma_\omega$ or of some $\omega\in\mathfrak{D}(\Sigma)\setminus\overline{\Psi}(\mathrm{T}')$.

First, we consider the case that there exists an $\omega\in\mathfrak{F}(\Sigma)$ with $\mathrm{ex}_{\Sigma'}(\sigma)\preccurlyeq\omega$. Then, there is a unique $\eta\in\Sigma_n$ with $\omega\preccurlyeq\eta$, and hence it holds $\Sigma_\omega=\{\omega,\eta\}$. Moreover, it holds $\overline{\Psi}(\mathrm{T}')_\omega\not\subseteq\Sigma_\omega$, and thus there is a $\vartheta\in\mathrm{T}'_{n-1}\setminus\mathrm{T}$ with $\omega\preccurlyeq\psi(\vartheta)$ and hence $\mathrm{ex}_{\Sigma'}(\sigma)\preccurlyeq\psi(\vartheta)$. Then, the claim follows from 3.6.14.

Now, we consider the case that $\mathrm{ex}_{\Sigma'}(\sigma)$ is not a face of some $\omega\in\mathfrak{F}(\Sigma)$. As Σ is tightly separable over Σ', we have $\mathrm{ex}_{\Sigma'}(\sigma)\subseteq\mathrm{fr}(|\Sigma|)$, and by 2.3.17 and our hypotheses there is an $\omega\in\mathfrak{D}(\Sigma)\cap\overline{\Psi}(\mathrm{T}')$ with $\mathrm{ex}_{\Sigma'}(\sigma)\preccurlyeq\omega$. Equifulldimensionality of T' implies the existence of a $\tau\in\mathrm{T}'_{n-1}$ with $\omega\preccurlyeq\psi(\tau)$ and hence $\mathrm{ex}_{\Sigma'}(\sigma)\preccurlyeq\psi(\tau)$. Moreover, it is clear that $\tau\notin\mathrm{T}$, and then the claim follows from 3.6.14. □

We end this section with some further technical results used for the recursive construction of packings in 3.7.1. They all state what happens if we extend Σ by some pullback $\Sigma(\mathrm{T}')$. The first one (3.6.16) states essentially that only cones are added that have at least one 1-dimensional face in Σ. The second one (3.6.17) states essentially that if T' is complete, then the 1-dimensional cone ξ – and *only* this 1-dimensional cone – is "packed". Its corollary 3.6.18 finally shows that only full cones (and their faces) are added.

(3.6.16) Proposition *Let $\Sigma' \subseteq \Sigma$ be a subfan with $\xi \in \Sigma'_1$ and $\Sigma_{\max} \subseteq \bigcup_{\rho\in\Sigma'_1} \Sigma_\rho$, let $\mathrm{T} \subseteq \mathrm{T}'$ be a relatively simplicial, separable W_ξ-extension, and let (q, a, B) be a W-pullback datum along ξ that is good for Σ and T'. Then, it holds*

$$\Sigma_{q,a,B}(\mathrm{T}')_{\max} \subseteq \bigcup_{\rho\in\Sigma'_1} \Sigma_{q,a,B}(\mathrm{T}')_\rho.$$

Proof. Since $\Psi(\mathrm{T}') \subseteq \Sigma(\mathrm{T}')_\xi$ by 3.6.3 c) and $\xi \in \Sigma'_1$, it follows from 2.1.11 a) and 3.6.12 a) that

$$\Sigma(\mathrm{T}')_{\max} \subseteq \Sigma_{\max} \cup \overline{\Psi}(\mathrm{T}')_{\max} \subseteq \Big(\bigcup_{\rho\in\Sigma'_1} \Sigma_\rho\Big) \cup \Sigma(\mathrm{T}')_\xi \subseteq \bigcup_{\rho\in\Sigma'_1} \Sigma(\mathrm{T}')_\rho. \qquad \square$$

(3.6.17) Proposition *Let $\Sigma' \subseteq \Sigma$ be a subfan, let $\mathrm{T} \subseteq \mathrm{T}'$ be a relatively simplicial, separable W_ξ-extension, and let (q, a, B) be a W-pullback datum along ξ that is good for Σ and T'.*

a) If T' is equifulldimensional, then it holds

$$C(\Sigma', \Sigma) \setminus \{\xi\} \subseteq C(\Sigma', \Sigma_{q,a,B}(\mathrm{T}')).$$

b) If T' is complete, then it holds

$$C(\Sigma', \Sigma) \setminus \{\xi\} = C(\Sigma', \Sigma_{q,a,B}(\mathrm{T}')).$$

Proof. a) Let $\rho \in C(\Sigma', \Sigma) \setminus \{\xi\}$. Then it holds $\rho \in \Sigma'_1$, and by 3.3.1 and 2.3.17 there exists a $\sigma \in \mathfrak{D}(\Sigma) \cup \mathfrak{F}(\Sigma)$ with $\rho \preccurlyeq \sigma$. We have to show that $\Sigma(\mathrm{T}')/\rho$ is not complete, that is, again by 3.3.1 and 2.3.17, that there exists an $\omega \in \mathfrak{D}(\Sigma(\mathrm{T}')) \cup \mathfrak{F}(\Sigma(\mathrm{T}'))$ with $\rho \preccurlyeq \omega$. If $\sigma \notin \overline{\Psi}(\mathrm{T}')$, then 3.6.12 c) implies $\sigma \in \mathfrak{D}(\Sigma(\mathrm{T}')) \cup \mathfrak{F}(\Sigma(\mathrm{T}'))$ and therefore the claim. So, suppose that $\sigma \in \overline{\Psi}(\mathrm{T}')$.

We set $\mathbb{A} := \{\tau \in \mathrm{T}'_{n-1} \mid \sigma \preccurlyeq \psi(\tau)\}$. If $\tau \in \mathbb{A}$, then we set

$$\mathbb{A}_\tau := \{\vartheta \in \mathrm{face}(\psi(\tau))_{n-1} \mid \rho \preccurlyeq \vartheta\},$$

and on use of 1.4.19 we see that $\rho = \bigcap \mathbb{A}_\tau$. If there are $\tau \in \mathbb{A}$ and $\vartheta \in \mathbb{A}_\tau$ with $\vartheta \in \mathfrak{F}(\Sigma(\mathrm{T}'))$, then the claim is proven. So, we assume that for every $\tau \in \mathbb{A}$ and every $\vartheta \in \mathbb{A}_\tau$ it holds $\vartheta \notin \mathfrak{F}(\Sigma(\mathrm{T}'))$, and we show that this leads to a contradiction.

First, 3.6.12 c) implies that for every $\tau \in \mathbb{A}$ and every $\vartheta \in \mathbb{A}_\tau$ there exists $\eta^{(\tau,\vartheta)} \in \Sigma(\mathrm{T}')_n$ with $\vartheta = \eta^{(\tau,\vartheta)} \cap \psi(\tau)$. If there is a $\tau \in \mathbb{A}$ such that for every $\vartheta \in \mathbb{A}_\tau$ it holds $\eta^{(\tau,\vartheta)} \in \overline{\Psi}(\mathrm{T}')$, then it follows $\eta^{(\tau,\vartheta)} \in \Psi(\mathrm{T}')$ and hence $\xi \preccurlyeq \eta^{(\tau,\vartheta)}$ for every $\vartheta \in \mathbb{A}_\tau$, yielding $\xi \preccurlyeq \bigcap \mathbb{A}_\tau = \rho$ and thus the contradiction $\xi = \rho$. Therefore, for every $\tau \in \mathbb{A}$ there exists a $\vartheta^{(\tau)} \in \mathbb{A}_\tau$ with $\eta^{(\tau,\vartheta^{(\tau)})} \in \Sigma \setminus \overline{\Psi}(\mathrm{T}')$.

Next, let $\tau \in \mathbb{A}$. If $\xi \preccurlyeq \vartheta^{(\tau)}$, then we have $\xi \preccurlyeq \vartheta^{(\tau)} \preccurlyeq \eta^{(\tau,\vartheta^{(\tau)})}$ and hence get the contradiction $\eta^{(\tau,\vartheta^{(\tau)})} \in \Sigma_\xi \subseteq \overline{\Psi}(\mathrm{T}')$. So, it holds $\xi \not\preccurlyeq \vartheta^{(\tau)}$ and therefore $\vartheta^{(\tau)} \neq \psi(\mathrm{in}_\mathrm{T}(\tau))$. Moreover, since (q, a, B) is good for Σ and T' it holds

$$\vartheta^{(\tau)} = \eta^{(\tau,\vartheta^{(\tau)})} \cap \psi(\tau) = \eta^{(\tau,\vartheta^{(\tau)})} \cap \psi(\mathrm{in}_\mathrm{T}(\tau)) \preccurlyeq \psi(\mathrm{in}_\mathrm{T}(\tau)) \preccurlyeq \psi(\tau).$$

As $\dim(\vartheta^{(\tau)}) = n - 1$ and $\dim(\psi(\tau)) = n$ it follows from the above that $\psi(\mathrm{in}_\mathrm{T}(\tau)) = \psi(\tau)$ and hence $\tau \in \mathrm{T}$. This shows that $\mathbb{A} \subseteq \mathrm{T}$.

Finally, equifulldimensionality of T' and 3.6.12 b) imply $\mathbb{A} \neq \emptyset$. From this we get $\sigma \notin \mathfrak{D}(\Sigma)$, hence $\sigma \in \mathfrak{F}(\Sigma)$. But since $\mathbb{A} \subseteq \mathrm{T}$ it holds $\overline{\Psi}(\mathrm{T}')_\sigma \subseteq \Sigma_\sigma$, and thus 3.6.12 c) implies the contradiction $\sigma \in \mathfrak{F}(\Sigma(\mathrm{T}'))$.

b) Let $\rho \in C(\Sigma', \Sigma(\mathrm{T}'))$. It holds $\Sigma/\rho \subseteq \Sigma(\mathrm{T}')/\rho$, and $\Sigma(\mathrm{T}')/\rho$ is not complete. Hence, Σ/ρ is not complete, too, and therefore we have $\rho \in C(\Sigma', \Sigma)$. Moreover, as $\Sigma_\xi \subseteq \overline{\Psi}(\mathrm{T}')$ we have $\Sigma(\mathrm{T}')_\xi = \overline{\Psi}(\mathrm{T}')_\xi$, and then 3.6.6 implies that $\Sigma(\mathrm{T}')/\xi = \overline{\Psi}(\mathrm{T}')/\xi = \mathrm{T}'$ is complete. Thus, it holds $\rho \neq \xi$, and therefore the claim follows from a). □

(3.6.18) Corollary *Let $\Sigma' \subseteq \Sigma$ be a subfan, let $\mathrm{T} \subseteq \mathrm{T}'$ be a relatively simplicial, separable W_ξ-completion, and let (q, a, B) be a W-pullback datum along ξ that is good for Σ and T'. If every maximal element of the ordered set $\bigcup_{\rho \in \Sigma'_1 \setminus C(\Sigma',\Sigma)} \Sigma_\rho$ is full, then so is every maximal element of the ordered set $\bigcup_{\rho \in \Sigma'_1 \setminus C(\Sigma',\Sigma_{q,a,B}(\mathrm{T}'))} \Sigma_{q,a,B}(\mathrm{T}')_\rho$.*

Proof. Let σ be a maximal element of $\bigcup_{\rho \in \Sigma'_1 \setminus C(\Sigma',\Sigma(\mathrm{T}'))} \Sigma(\mathrm{T}')_\rho$. Then, there is a $\rho \in \Sigma'_1 \setminus C(\Sigma', \Sigma(\mathrm{T}'))$ with $\rho \preccurlyeq \sigma$, and on use of 2.1.11 a) and 3.6.12 a) it moreover holds

$$\sigma \in \Sigma(\mathrm{T}')_{\max} \subseteq (\Sigma_{\max} \setminus \overline{\Psi}(\mathrm{T}')) \cup \{\psi(\tau) \mid \tau \in \mathrm{T}'_{\max}\}.$$

If $\sigma = \psi(\tau)$ for some $\tau \in \mathrm{T}'_{\max}$, then equifulldimensionality of T' and 3.6.3 d) imply that σ is full. So, suppose that $\sigma \in \Sigma_{\max} \setminus \overline{\Psi}(\mathrm{T}')$ and hence $\sigma \in \Sigma_\rho \setminus \Sigma_\xi$. Then, we get $\rho \neq \xi$ and hence $\rho \in \Sigma'_1 \setminus C(\Sigma', \Sigma)$ on use of 3.6.17 a). Therefore, σ is a maximal element of $\bigcup_{\rho \in \Sigma'_1 \setminus C(\Sigma',\Sigma)} \Sigma_\rho$ and thus full. □

3.7. Existence of packings and completions

Finally we can harvest the fruits of the above preparations and thus prove the existence of strong completions.

(3.7.1) Proposition *Let $n > 1$, and let Σ be a W-fan in V with $\Sigma_1 \neq \emptyset$. Moreover, suppose that for every $\mathbb{R}$-vector space V' with $\dim(V') < n$, every R-structure W' on V' and every W'-fan Σ' in V' there exists a strong W'-completion of Σ'. Then, there exists an increasing sequence $(\Sigma^{(i)})_{i \in \mathbb{N}_0}$ of relatively simplicial, tightly separable W-extensions of Σ such that the sequence $(c(\Sigma, \Sigma^{(i)}))_{i \in \mathbb{N}_0}$ is decreasing and that for every $i \in \mathbb{N}_0$ the following statements hold:*

i) If $c(\Sigma, \Sigma^{(i)}) \neq 0$, then $c(\Sigma, \Sigma^{(i)}) > c(\Sigma, \Sigma^{(i+1)})$;

ii) $\Sigma^{(i)}_{\max} \subseteq \bigcup_{\rho\in\Sigma_1} \Sigma^{(i)}_\rho$;

iii) Every maximal element of $\bigcup_{\rho\in\Sigma_1\setminus C(\Sigma,\Sigma^{(i)})} \Sigma^{(i)}_\rho$ *is full.*

Proof. Without loss of generality we may assume that $R = K$ by 1.4.1, and then we show the existence of such a sequence by recursion. First, we may set $\Sigma^{(0)} := \Sigma$. Indeed, Σ is a relatively simplicial, tightly separable W-extension of Σ by 3.1.6 and 3.2.3. Moreover, as $\Sigma_1 \neq \emptyset$ we have $\Sigma_{\max} \subseteq \bigcup_{\rho\in\Sigma_1} \Sigma_\rho$ by 1.4.17. Finally, let σ be a maximal element of $\bigcup_{\rho\in\Sigma_1\setminus C(\Sigma,\Sigma)} \Sigma_\rho$. Then, there is a $\rho \in \Sigma_1\setminus C(\Sigma,\Sigma)$ with $\rho \preccurlyeq \sigma$, and then σ is clearly a maximal element of Σ_ρ. Therefore, by 2.2.6 we see that σ/ρ is a maximal element of the complete W_ρ-fan Σ/ρ and hence full in V_ρ by 2.3.16. Thus, σ is full in V by 2.2.7 b).

Now, let $i \in \mathbb{N}_0$, and suppose there is an increasing sequence $(\Sigma^{(j)})^i_{j=0}$ of relatively simplicial, tightly separable W-extensions of Σ such that the sequence $(c(\Sigma,\Sigma^{(j)}))^i_{j=0}$ is decreasing, that for every $j \in [0, i-1]$ with $c(\Sigma,\Sigma^{(j)}) \neq 0$ it holds $c(\Sigma,\Sigma^{(j)}) > c(\Sigma,\Sigma^{(j+1)})$, and that for every $j \in [0, i]$ every maximal element of $\Sigma^{(j)}$ is contained in $\bigcup_{\rho\in\Sigma_1} \Sigma^{(j)}_\rho$ and every maximal element of $\bigcup_{\rho\in\Sigma_1\setminus C(\Sigma,\Sigma^{(j)})} \Sigma^{(j)}_\rho$ is full.

If $c(\Sigma,\Sigma^{(i)}) = 0$, then we may obviously set $\Sigma^{(i+1)} := \Sigma^{(i)}$. So, suppose that $c(\Sigma,\Sigma^{(i)}) \neq 0$. Hence, there is a $\xi \in C(\Sigma,\Sigma^{(i)})$, and then the W_ξ-fan $\mathrm{T} := \Sigma^{(i)}/\xi$ in V_ξ is not complete. By hypothesis there exists a strong W_ξ-completion $(\overline{\mathrm{T}}, \widehat{\mathrm{T}})$ of T, and from 3.3.10 we know that $\widehat{\mathrm{T}}$ is relatively simplicial and separable over T. By 3.6.9 there exists a W-pullback datum (q, a, B) along ξ that is very good for $\Sigma^{(i)}$ and $\widehat{\mathrm{T}}$. We set $\Sigma^{(i+1)} := \Sigma^{(i)}_{q,a,B}(\widehat{\mathrm{T}})$. Then, it follows from 3.6.11, 3.6.15, 3.6.16, 3.6.17 b) and 3.6.18 that $\Sigma^{(i+1)}$ is a W-fan as desired. □

(3.7.2) Corollary *Let $n > 1$, and let Σ be a W-fan in V with $\Sigma_1 \neq \emptyset$. Moreover, suppose that for every $\mathbb{R}$-vector space V' with $\dim(V') < n$, every R-structure W' on V' and every W'-fan Σ' in V' there exists a strong W'-completion of Σ'. Then, there exists a W-packing of Σ.*

Proof. Let $(\Sigma^{(i)})_{i\in\mathbb{N}_0}$ be a sequence as is proved to exist in 3.7.1. Then, there is an $i \in \mathbb{N}_0$ such that $c(\Sigma,\Sigma^{(i)}) = 0$, and $\Sigma^{(i)}$ is a relatively simplicial, tightly separable W-extension of Σ. Moreover, it is a quasipacking of Σ by 3.3.4. Since every maximal element of $\Sigma^{(i)}$ is a maximal element of $\bigcup_{\rho\in\Sigma_1\setminus C(\Sigma,\Sigma^{(i)})} \Sigma^{(i)}_\rho$, the fan $\Sigma^{(i)}$ is equifulldimensional and thus a W-packing of Σ. □

(3.7.3) Proposition *Let Σ be a W-fan in V with $\Sigma_1 \neq \emptyset$. If there exists a W-packing of Σ, then there exists a strong W-completion of Σ.*

Proof. Let $\overline{\Sigma}$ be a W-packing of Σ which is equifulldimensional since $\Sigma_1 \neq \emptyset$. If $\overline{\Sigma}$ is complete, then $(\overline{\Sigma}, \overline{\Sigma})$ is a strong W-completion of Σ by

3.3.11. So, suppose that $\overline{\Sigma}$ is not complete. By 3.4.5 we can choose a W-extension $\overline{\Sigma} \subseteq \overline{\Sigma}'$ that is relatively simplicial over Σ and a family $H = (H_{\sigma,\tau})_{(\sigma,\tau)\in\mathfrak{F}(\Sigma')^2}$ of linear W-hyperplanes in V such that if $\sigma,\tau \in \mathfrak{F}(\Sigma')$, then $H_{\sigma,\tau}$ separates σ and τ in their intersection, subject to the conditions that the complete W-semifan $\Omega := \overline{\Omega}_{\overline{\Sigma}',H}$ associated with $\overline{\Sigma}'$ and H is a fan, and that $\overline{\Sigma}'$ is complete or $\mathfrak{F}(\overline{\Sigma}) \subseteq \mathfrak{F}(\overline{\Sigma}')$. If $\overline{\Sigma}'$ is complete, then $(\overline{\Sigma}, \overline{\Sigma}')$ is obviously a strong W-completion of Σ. So, suppose that $\overline{\Sigma}'$ is not complete, and hence $\mathfrak{F}(\overline{\Sigma}) \subseteq \mathfrak{F}(\overline{\Sigma}')$.

We set

$$\mathrm{T} := \{\sigma \in \Omega \mid \sigma \subseteq \mathrm{cl}(V \setminus |\overline{\Sigma}|)\} \text{ and } \mathrm{T}' := \{\sigma \in \Omega \mid \sigma \subseteq \mathrm{fr}(|\overline{\Sigma}|)\}.$$

By 3.4.6 and 2.3.16 we see that T is a W-fan with $|\overline{\Sigma}| \cup |\mathrm{T}| = V$ and with $|\overline{\Sigma}| \cap |\mathrm{T}| = |\overline{\mathfrak{F}}(\overline{\Sigma})|$, and that T' is a subfan of T and a W-subdivision of the subfan $\overline{\mathfrak{F}}(\overline{\Sigma})$ of $\overline{\Sigma}$. By 2.4.8 there exists a simplicial W-subdivision $\overline{\mathrm{T}}$ of T, and we denote by $\overline{\mathrm{T}}'$ the subdivision of T' induced by $\overline{\mathrm{T}}$. By 2.4.5 and 2.4.1 this is a simplicial W-subdivision of $\overline{\mathfrak{F}}(\overline{\Sigma})$.

Now, we set $\widetilde{\Sigma} := \mathrm{adj}_{\Sigma}(\overline{\Sigma}, \overline{\mathrm{T}}')$ and $\widehat{\Sigma} := \widetilde{\Sigma} \cup \overline{\mathrm{T}}$. As $\overline{\mathfrak{F}}(\overline{\Sigma}) = \mathrm{ex}_{\Sigma}(\overline{\Sigma})$ by 3.3.9, it follows from 3.5.6 c) that $\widetilde{\Sigma}$ is a W-packing of Σ. Moreover, it follows from 3.5.7 that $\widehat{\Sigma}$ is a W-extension of $\widetilde{\Sigma}$ that is relatively simplicial over Σ and that $|\widehat{\Sigma}| = |\overline{\Sigma}| \cup |\overline{\mathrm{T}}| = V$. Hence, $\widehat{\Sigma}$ is complete, and thus $(\overline{\Sigma}, \widehat{\Sigma})$ is a strong W-completion of Σ as desired. □

(3.7.4) Theorem *Every W-fan in V has a strong W-completion.*

Proof. Let Σ be a W-fan in V. If $\Sigma_1 = \emptyset$, then Σ has a strong W-completion by 2.2.14 and 3.3.12. So, we may suppose that $\Sigma_1 \neq \emptyset$ and hence $n \geq 1$, and then we show the claim by induction on n. If $n = 1$, then it holds by 3.3.11 and 3.3.13. So, we suppose that $n > 1$ and that the claim holds for strictly smaller values of n. Then, 3.7.2 implies the existence of a W-packing of Σ, and then 3.7.3 yields the existence of a strong W-completion of Σ. Thus, the claim is proven. □

(3.7.5) Corollary *Every (simplicial) W-semifan in V has a (simplicial) W-completion.*

Proof. It follows from 3.7.4 and 3.3.10 that every W-fan in V has a relatively simplicial, separable W-completion, and hence 2.2.12 and 3.2.7 imply the claim. □

4. The combinatorics of a fan

4.1. The Picard group of a fan

Let $R \subseteq \mathbb{R}$ be a subring, let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim(V)$, let W be an R-structure on V, and let Σ be a W-fan in V.

In this section we introduce the notions of virtual polytopes and of Picard group of a fan, based on [12].

(4.1.1) We denote by $\Delta_{W,\Sigma} : W^* \to (W^*)^\Sigma$ the diagonal morphism in $\mathsf{Mod}(R)$, and we denote by $P_{W,\Sigma}$ the set of families $(m_\sigma + \sigma)_{\sigma\in\Sigma}$ of subsets of V^* with $(m_\sigma)_{\sigma\in\Sigma} \in (W^*)^\Sigma$. Then, there is a surjective map

$$h_{W,\Sigma} : (W^*)^\Sigma \twoheadrightarrow P_{W,\Sigma},\ (m_\sigma)_{\sigma\in\Sigma} \mapsto (m_\sigma + \sigma^\vee)_{\sigma\in\Sigma}.$$

For families $(m_\sigma)_{\sigma\in\Sigma}$ and $(m'_\sigma)_{\sigma\in\Sigma}$ in W^* we have

$$h_{W,\Sigma}((m_\sigma)_{\sigma\in\Sigma}) = h_{W,\Sigma}((m'_\sigma)_{\sigma\in\Sigma})$$

if and only if $(m_\sigma - m'_\sigma)_{\sigma\in\Sigma} \in \prod_{\sigma\in\Sigma} \sigma^\perp$, and hence there is a canonical structure of R-module on $P_{W,\Sigma}$ such that $h_{W,\Sigma}$ is an epimorphism in $\mathsf{Mod}(R)$. We always furnish $P_{W,\Sigma}$ with this structure, and we denote by $K_{W,\Sigma}$ the image of $h_{W,\Sigma} \circ \Delta_{W,\Sigma}$.

Clearly, $\Delta_{V,\Sigma}$ is rational with respect to the R-structures W^* and $(W^*)^\Sigma$. Moreover, $P_{W,\Sigma}$ is an R-structure on $P_{V,\Sigma}$ and induces the R-structure $K_{W,\Sigma}$ on $K_{V,\Sigma}$, and $h_{V,\Sigma}$ is rational with respect to $(W^*)^\Sigma$ and $P_{W,\Sigma}$.

If no confusion about W can arise we write Δ_Σ, P_Σ, h_Σ and K_Σ instead of $\Delta_{W,\Sigma}$, $P_{W,\Sigma}$, $h_{W,\Sigma}$ and $K_{W,\Sigma}$.

(4.1.2) It is clear that $\Delta_{W,\Sigma}$ is a monomorphism if and only if $\Sigma \neq \emptyset$ or $n = 0$. Moreover, we have $\mathrm{Ker}(h_{W,\Sigma}) = \prod_{\sigma\in\Sigma}(W^* \cap \sigma^\perp)$ by 4.1.1. Thus, $h_{W,\Sigma}$ is a monomorphism if and only if $\Sigma = \emptyset$ or $n = 0$. Indeed, we have $\prod_{\sigma\in\Sigma}(W^* \cap \sigma^\perp) = 0$ if and only if $\prod_{\sigma\in\Sigma} \sigma^\perp = 0$, hence if and only if $\sigma^\perp = 0$ for every $\sigma \in \Sigma$, and therefore if and only if $\Sigma = \emptyset$ or $n = 0$.

The above and 1.1.5 show that $\mathrm{Ker}(h_{W,\Sigma} \circ \Delta_{W,\Sigma}) = W^* \cap \langle\Sigma\rangle^\perp$. Hence, W-rationality of $\langle\Sigma\rangle$ implies that $h_{W,\Sigma} \circ \Delta_{W,\Sigma}$ is a monomorphism if and only if Σ is full.

(4.1.3) It holds $K_{W,\Sigma} = 0$ if and only if $\Sigma_1 = \emptyset$. Indeed, $K_{W,\Sigma} = 0$ is equivalent to $W^* = \mathrm{Ker}(h_{W,\Sigma} \circ \Delta_{W^*})$, and by 4.1.2 this holds if and only if $W^* = W^* \cap \langle\Sigma\rangle^\perp$, that is, if and only if $\Sigma_1 = \emptyset$.

(4.1.4) Suppose that Σ is full. Then, $h_{W,\Sigma} \circ \Delta_{W,\Sigma}$ is a monomorphism by 4.1.2 and hence induces an isomorphism $W^* \xrightarrow{\cong} K_{W,\Sigma}$ in $\mathsf{Mod}(R)$ by coastriction. Its inverse is denoted by $f_{W,\Sigma} : K_{W,\Sigma} \xrightarrow{\cong} W^*$. Clearly, $f_{V,\Sigma}$ is rational with respect to $K_{W,\Sigma}$ and W^*.

If no confusion about W can arise we write f_Σ instead of $f_{W,\Sigma}$.

(4.1.5) If $(m_\sigma)_{\sigma\in\Sigma}$ and $(m'_\sigma)_{\sigma\in\Sigma}$ are families in W^* with $m_\sigma+\sigma^\vee = m'_\sigma+\sigma^\vee$ for every $\sigma \in \Sigma$, then it is readily checked that for $\sigma \in \Sigma$ and $\tau \preccurlyeq \sigma$ the relations $m_\sigma - m_\tau \in \tau^\perp$ and $m'_\sigma - m'_\tau \in \tau^\perp$ are equivalent. This allows to define a *virtual polytope over Σ (with respect to W)* to be an element $(m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} \in P_{W,\Sigma}$ such that for every $\sigma \in \Sigma$ and every $\tau \preccurlyeq \sigma$ it holds $m_\sigma - m_\tau \in \tau^\perp$.

We denote by $\overline{P}_{W,\Sigma}$ the set of virtual polytopes over Σ with respect to W. It is easy to see that $\overline{P}_{W,\Sigma}$ is a sub-R-module of $P_{W,\Sigma}$ containing $K_{W,\Sigma}$. We denote by $g_{W,\Sigma} : K_{W,\Sigma} \hookrightarrow \overline{P}_{W,\Sigma}$ the canonical injection and by $e_{W,\Sigma} : \overline{P}_{W,\Sigma} \twoheadrightarrow \mathrm{Pic}_W(\Sigma)$ its cokernel in $\mathsf{Mod}(R)$. The R-module $\mathrm{Pic}_W(\Sigma)$ is called *the Picard module of Σ (with respect to W)*, and the exact sequence

$$0 \longrightarrow K_{W,\Sigma} \xrightarrow{g_{W,\Sigma}} \overline{P}_{W,\Sigma} \xrightarrow{e_{W,\Sigma}} \mathrm{Pic}_W(\Sigma) \longrightarrow 0$$

in $\mathsf{Mod}(R)$ is denoted by $\mathbb{T}_{W,\Sigma}$ and called *the upper standard sequence of Σ (with respect to W).*

For $p = (m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} \in P_{W,\Sigma}$ it holds $p \in \overline{P}_{W,\Sigma}$ if and only if for every $\sigma \in \Sigma$ and every $\rho \in \sigma_1$ it holds $m_\sigma - m_\rho \in \rho^\perp$. Indeed, if this condition holds and $\sigma \in \Sigma$ and $\tau \preccurlyeq \sigma$, then we have $m_\sigma - m_\rho \in \rho^\perp$ and $m_\tau - m_\rho \in \rho^\perp$ for every $\rho \in \tau_1$, hence $m_\sigma - m_\tau = (m_\sigma - m - \rho) - (m_\tau - m_\rho) \in \rho^\perp$ for every $\rho \in \tau_1$, and thus $m_\sigma - m_\tau \in \bigcap_{\rho\in\tau_1} \rho^\perp = \tau^\perp$.

Clearly, $\overline{P}_{W,\Sigma}$ is an R-structure on $\overline{P}_{V,\Sigma}$, and $g_{V,\Sigma}$ is rational with respect to $K_{W,\Sigma}$ and $\overline{P}_{W,\Sigma}$.

If no confusion about W can arise we write $\overline{P}_\Sigma$, g_Σ, $\mathrm{Pic}(\Sigma)$, e_Σ and $\mathbb{T}_\Sigma$ instead of $\overline{P}_{W,\Sigma}$, $g_{W,\Sigma}$, $\mathrm{Pic}_W(\Sigma)$, $e_{W,\Sigma}$ and $\mathbb{T}_{W,\Sigma}$.

(4.1.6) Proposition *Suppose that R is a principal ideal domain and that Σ is fulldimensional. Then, $\mathrm{Pic}_W(\Sigma)$ is free.*

Proof. Since Σ is fulldimensional, there exists an $\omega \in \Sigma_n$. Now, let $(m_\sigma)_{\sigma\in\Sigma} \in (W^*)^\Sigma$ and $r \in R \setminus 0$ such that there exists an $m \in W^*$ with $(rm_\sigma + \sigma^\vee)_{\sigma\in\Sigma} = (m + \sigma^\vee)_{\sigma\in\Sigma}$. Then, it holds $rm_\omega - m \in \omega^\perp = 0$ and hence $(rm_\sigma + \sigma^\vee)_{\sigma\in\Sigma} = (rm_\omega + \sigma^\vee)_{\sigma\in\Sigma}$. This implies $r(m_\sigma - m_\omega) \in \sigma^\perp$ and hence $m_\sigma - m_\omega \in \sigma^\perp$ for every $\sigma \in \Sigma$, and from this it follows $(m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} = (m_\omega + \sigma^\vee)_{\sigma\in\Sigma} \in K_{W,\Sigma}$. Thus, $\mathrm{Pic}_W(\Sigma)$ is torsionfree, and since it is finitely generated and as R is a principal ideal domain, this proves that $\mathrm{Pic}_W(\Sigma)$ is free. □

(4.1.7) The group $\mathrm{Pic}_W(\Sigma)$ is not necessarily free, also if Σ is supposed to be full and $R = \mathbb{Z}$. A counterexample is given by the full $\mathbb{Z}^2$-fan

$$\Sigma = \{0, \mathrm{cone}((1,0)), \mathrm{cone}((1,2))\}$$

in $\mathbb{R}^2$ with $\mathrm{Pic}_{\mathbb{Z}^2}(\Sigma) \cong \mathbb{Z}/2\mathbb{Z}$.

Furthermore, the converse of 4.1.6 does not hold, also if Σ is supposed to be full and $R = \mathbb{Z}$. Indeed, the full $\mathbb{Z}^2$-fan

$$\Sigma = \{0, \mathrm{cone}((1,0)), \mathrm{cone}((0,1)), \mathrm{cone}((-1,0))\}$$

in $\mathbb{R}^2$ is not fulldimensional, but it is readily checked that $\mathrm{Pic}_{\mathbb{Z}^2}(\Sigma) \cong \mathbb{Z}$.

The rest of this section is devoted to preparing the ground for characterising regularity (and in particular simpliciality) of a fan in terms of its Picard group in 4.2.8 and 4.2.9.

(4.1.8) Proposition *Let $\omega \in \Sigma$, let $(m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} \in \overline{P}_{W,\Sigma}$, and let $m'_\sigma := m_\omega$ for every $\sigma \preccurlyeq \omega$ and $m'_\sigma := m_\sigma$ for every $\sigma \in \Sigma \setminus \mathrm{face}(\omega)$. Then, it holds $(m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} = (m'_\sigma + \sigma^\vee)_{\sigma\in\Sigma}$.*

Proof. For $\sigma \preccurlyeq \omega$ it holds $m'_\sigma - m_\sigma = m_\omega - m_\sigma \in \sigma^\perp$ and hence $m'_\sigma + \sigma^\vee = m_\sigma + \sigma^\vee$. The claim follows from this. □

(4.1.9) Corollary *Let $\omega \in \Sigma$, and let $p \in \mathrm{Pic}_W(\Sigma)$. Then, there exists a unique $(m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} \in \overline{P}_{W,\Sigma}$ such that $e_{W,\Sigma}((m_\sigma + \sigma^\vee)_{\sigma\in\Sigma}) = p$ and that $m_\sigma = 0$ for every $\sigma \in \mathrm{face}(\omega)$.*

Proof. This follows immediately from the definition of $\mathrm{Pic}_W(\Sigma)$ and 4.1.8. □

(4.1.10) Corollary *Let σ be a sharp W-polycone in V. Then, it holds $\mathrm{Pic}_W(\mathrm{face}(\sigma)) = 0$.*

Proof. Clear from 4.1.8. □

(4.1.11) Corollary *Let σ be a sharp W-polycone in V, and let $(m_\rho)_{\rho\in\sigma_1}$ be a family in W^*. Then, the following statements are equivalent:*

(i) There is a family $(m_\tau)_{\tau\in\mathrm{face}(\sigma)\setminus\sigma_1}$ in W^* such that $(m_\tau + \tau^\vee)_{\tau\preccurlyeq\sigma} \in \overline{P}_{W,\mathrm{face}(\sigma)}$;

(ii) It holds $W^* \cap (\bigcap_{\rho\in\sigma_1}(m_\rho + \rho^\perp)) \neq \emptyset$;

(iii) It holds $\bigcap_{\rho\in\sigma_1}(m_\rho + \rho^\perp) \neq \emptyset$.

Proof. From 4.1.10 it follows that (i) holds if and only if there exists an $m \in W^*$ such that $m - m_\rho \in \rho^\perp$ for every $\rho \in \sigma_1$, and hence it is equivalent to (ii). Moreover, (ii) and (iii) are equivalent since a W^*-rational affine sub-$\mathbb{R}$-space of V is nonempty if and only if it meets W^*. □

(4.1.12) Lemma *Let A be a ring, let L be a free A-module of finite rank, let $c_L : L \xrightarrow{\cong} L^{**}$ denote the canonical isomorphism in $\mathsf{Mod}(A)$, and let E be a generating set of L. Then, E is a basis of L if and only if for every family $(u_e)_{e\in E}$ in L^* it holds $\bigcap_{e\in E}(u_e + \mathrm{Ker}(c_L(e))) \neq \emptyset$.*

Proof. Suppose that E is a basis of L, and let $(u_e)_{e\in E}$ be a family in L^*. For $e \in E$ we denote by e^* the element of the dual basis of E corresponding to e. Then, for every $e \in E$ there is a family $(u_e^{(f)})_{f\in E}$ in A with $u_e = \sum_{f\in E} u_e^{(f)} f^*$, and it is readily checked that

$$\sum_{f\in E} u_f^{(f)} f^* \in \bigcap_{e\in E}(u_e + \mathrm{Ker}(c_L(e))).$$

Conversely, suppose the above condition to hold, and assume that E is not an A-basis of L, hence not free. There are $e_0 \in E$ and a family $(r_e)_{e\in E}$ in A with $r_{e_0} \neq 0$ such that, setting $F := E \setminus \{e_0\}$, it holds $r_{e_0}e_0 = \sum_{e\in F} r_e e$. The hypothesis implies that for every $u \in L^*$ there is a $v \in (u + \mathrm{Ker}(c_L(e_0))) \cap (\bigcap_{e\in F} \mathrm{Ker}(c_L(e)))$ and hence a $w \in \mathrm{Ker}(c_L(e_0))$ with $u = v - w \in (\bigcap_{e\in F} \mathrm{Ker}(c_L(e))) + \mathrm{Ker}(c_L(e_0))$. Therefore, it holds

$$L^* \subseteq (\textstyle\bigcap_{e\in F} \mathrm{Ker}(c_L(e))) + \mathrm{Ker}(c_L(e_0)) \subseteq$$

$$\mathrm{Ker}(c_L(\textstyle\sum_{e\in F} r_e e)) + \mathrm{Ker}(c_L(r_{e_0}e_0)) \subseteq \mathrm{Ker}(c_L(r_{e_0}e_0)),$$

hence $r_{e_0}e_0 = 0$ and thus the contradiction $e_0 = 0$. □

(4.1.13) Lemma *Let σ be a sharp W-polycone in V. Then, σ is W-regular if and only if for every family $(m_\rho)_{\rho\in\sigma_1}$ in W^* there is a family $(m_\tau)_{\tau\in\mathrm{face}(\sigma)\setminus\sigma_1}$ in W^* such that $(m_\tau + \tau^\vee)_{\tau\in\mathrm{face}(\sigma)} \in \overline{P}_{W,\mathrm{face}(\sigma)}$.*

Proof. First, suppose that σ is W-regular. Then, there exists a family $(a_\rho)_{\rho\in\sigma_1} \in \prod_{\rho\in\sigma_1}(W \cap \rho \setminus 0)$ and an R-basis E of W containing a_ρ for every $\rho \in \sigma_1$. Now, let $(m_\rho)_{\rho\in\sigma_1}$ be a family in W^*. For $\rho \in \sigma_1$ we set $m_{a_\rho} := m_\rho$, and for $e \in E \setminus \{a_\rho \mid \rho \in \sigma_1\}$ we set $m_e := 0 \in W^*$. Then, 4.1.12 implies

$$\emptyset \neq \bigcap_{e\in E}(m_e + e^\perp) \subseteq \bigcap_{\rho\in\sigma_1}(m_\rho + \rho^\perp),$$

and therefore the above condition follows from 4.1.11.

Conversely, suppose the above condition to hold. We set $V' := \langle\sigma\rangle$ and $W' := W \cap V'$, and we denote by $p : V \twoheadrightarrow V'$ the canonical epimorphism in $\mathsf{Mod}(\mathbb{R})$. Then, we may consider σ as a full W'-polycone in V', denoted by σ'. Since V' has a W-rational complement in V, it suffices to prove that σ' is W'-regular. So, let $(m'_\rho)_{\rho\in\sigma'_1}$ be a family in $(W')^*$. There is a family $(m_\rho)_{\rho\in\sigma_1}$ in W^* such that for every $\rho \in \sigma_1$ it holds $p(m_\rho) = m'_\rho$. Our hypothesis together with 4.1.11 implies that $\bigcap_{\rho\in\sigma_1}(m_\rho + \rho^{\perp,V})$ is nonempty, and hence we get

$$\emptyset \neq p\Big(\bigcap_{\rho\in\sigma_1}(m_\rho + \rho^{\perp,V})\Big) \subseteq \bigcap_{\rho\in\sigma'_1}(m'_\rho + \rho^{\perp,V'}).$$

Thus, it follows from 4.1.12 that σ' is W'-regular, and herewith the claim is proven. □

(4.1.14) Proposition *The following statements are equivalent:*

(i) Σ is W-regular;

(ii) For every family $(m_\rho)_{\rho\in\Sigma_1}$ in W^* there exists a family $(m_\sigma)_{\sigma\in\Sigma\setminus\Sigma_1}$ in W^* such that $(m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} \in \overline{P}_{W,\Sigma}$.

Proof. Suppose that Σ is W-regular, and let $(m_\rho)_{\rho\in\Sigma_1} \in (W^*)^{\Sigma_1}$. If $\sigma \in \Sigma$, then 4.1.13 implies the existence of a family $(m_\tau^{(\sigma)})_{\tau\preccurlyeq\sigma}$ in W^* with $(m_\rho^{(\sigma)})_{\rho\in\sigma_1} = (m_\rho)_{\rho\in\sigma_1}$ such that $(m_\tau^{(\sigma)} + \tau^\vee)_{\tau\preccurlyeq\sigma} \in \overline{P}_{W,\mathrm{face}(\sigma)}$. If $\tau \preccurlyeq \sigma \in \Sigma$, then it holds $m_\tau^{(\sigma)} - m_\tau^{(\tau)} = (m_\tau^{(\sigma)} - m_\rho) - (m_\tau^{(\tau)} - m_\rho) \in \rho^\perp$ for every $\rho \in \tau_1$, hence $m_\tau^{(\sigma)} - m_\tau^{(\tau)} \in \bigcap_{\rho\in\tau_1} \rho^\perp = \tau^\perp$ and therefore $m_\tau^{(\sigma)} + \tau^\vee = m_\tau^{(\tau)} + \tau^\vee$. This implies (ii).

Conversely, suppose that (ii) holds, let $\sigma \in \Sigma$, and let $(m_\rho)_{\rho\in\sigma_1} \in (W^*)^{\sigma_1}$. Setting $m_\rho := 0$ for every $\rho \in \Sigma_1 \setminus \sigma_1$, our hypothesis yields a family $(m_\tau)_{\tau\in\Sigma\setminus\Sigma_1}$ in W^* with $(m_\tau + \tau^\vee)_{\tau\in\Sigma} \in \overline{P}_{W,\Sigma}$, and from this we get $(m_\tau + \tau^\vee)_{\tau\preccurlyeq\sigma} \in \overline{P}_{W,\mathrm{face}(\sigma)}$. Then, 4.1.13 implies that σ is W-regular, and thus the claim is proven. □

4.2. The standard diagram of a fan

Let $R \subseteq \mathbb{R}$ be a subring, let V be an $\mathbb{R}$-vector space of finite dimension, let N be a $\mathbb{Z}$-structure on V, let $M := N^*$, and let Σ be an N-fan in V. Moreover, we consider $W := N \otimes_{\mathbb{Z}} R$ as an R-structure on V.

The aim in this section is to introduce, based on Cox's article [10] on homogeneous coordinate rings, further combinatorial data that are associated with a fan, and that will be crucial in our treatment of sheaves on toric schemes in IV.3. More precisely, these data are defined not by the fan Σ, but only by the set Σ_1 of its 1-dimensional cones. In fact, they depend on (rational) generators of these 1-dimensional cones. We will consider only fans that are rational with respect to some $\mathbb{Z}$-structure N and hence have a canonical choice of generators of their 1-dimensional cones, namely the minimal ones (see 1.4.33).

We start by defining a second exact standard sequence.

(4.2.1) By 1.4.33, every $\rho \in \Sigma_1$ has an N-minimal N-generator

$$\rho_N \in N \cap \rho \setminus 0 \subseteq W \cap \rho \setminus 0,$$

and by means of the canonical isomorphism $c_W : W \xrightarrow{\cong} W^{**}$ in $\mathsf{Mod}(R)$ we consider this as a morphism $W^* \to R$ in $\mathsf{Mod}(R)$. Hence, the set Σ_1 corresponds to a family $(\rho_N)_{\rho\in\Sigma_1}$ of morphisms $W^* \to R$ in $\mathsf{Mod}(R)$, and this family corresponds to a morphism

$$c_{N,R,\Sigma_1} : W^* \to R^{\Sigma_1},\ m \mapsto (\rho_N(m))_{\rho\in\Sigma_1}$$

in $\mathsf{Mod}(R)$. We denote by

$$a_{N,R,\Sigma_1} : R^{\Sigma_1} \twoheadrightarrow A_{N,R,\Sigma_1}$$

its cokernel. The exact sequence

$$W^* \xrightarrow{c_{N,R,\Sigma_1}} R^{\Sigma_1} \xrightarrow{a_{N,R,\Sigma_1}} A_{N,R,\Sigma_1} \longrightarrow 0$$

in $\mathsf{Mod}(R)$ is denoted by $\mathbb{S}_{N,R,\Sigma_1}$ and called *the lower standard sequence of* Σ_1 *(over R with respect to N).*

Composing the canonical projection $(W^*)^\Sigma \to (W^*)^{\Sigma_1}$ with the product $\prod_{\rho\in\Sigma_1} \rho_N : (W^*)^{\Sigma_1} \to R^{\Sigma_1}$ yields a morphism

$$d'_{N,R,\Sigma} : (W^*)^\Sigma \to R^{\Sigma_1}$$

in $\mathsf{Mod}(R)$ such that $d'_{N,R,\Sigma} \circ \Delta_{W,\Sigma} = c_{N,R,\Sigma_1}$.

Clearly, $c_{N,\mathbb{R},\Sigma_1}$ and $d'_{N,\mathbb{R},\Sigma_1}$ are rational with respect to the R-structures W^* or $(W^*)^\Sigma$, respectively, and R^{Σ_1}.

If $R = \mathbb{Z}$ and no confusion about N can arise we write c_{Σ_1}, A_{Σ_1}, a_{Σ_1}, $\mathbb{S}_{\Sigma_1}$ and d'_Σ instead of $c_{N,\mathbb{Z},\Sigma_1}$, $A_{N,\mathbb{Z},\Sigma_1}$, $a_{N,\mathbb{Z},\Sigma_1}$, $\mathbb{S}_{N,\mathbb{Z},\Sigma_1}$ and $d'_{N,\mathbb{Z},\Sigma}$.

(4.2.2) Example Let $k \in \mathbb{N}$. The lower standard sequence of the complete $\mathbb{Z}^2$-fan Σ in $\mathbb{R}^2$ with maximal cones

$$\operatorname{cone}((1,0),(0,1)), \operatorname{cone}((0,1),(-1,k)), \operatorname{cone}((-1,k),(0,-1)), \operatorname{cone}((0,-1),(1,0))$$

is

$$0 \to \mathbb{Z}^2 \xrightarrow{c_{\Sigma_1}} \mathbb{Z}^4 \xrightarrow{a_{\Sigma_1}} \mathbb{Z}^2 \to 0$$

with

$$c_{\Sigma_1} : \mathbb{Z}^2 \to \mathbb{Z}^4,\ (r,s) \mapsto (r, s, ks - r, -s)$$

and

$$a_{\Sigma_1} : \mathbb{Z}^4 \to \mathbb{Z}^2,\ (u_1,u_2,u_3,u_4) \mapsto (u_2 + u_4, u_1 + u_3 - ku_2).$$

(4.2.3) By 1.1.5 we have

$$\operatorname{Ker}(c_{N,R,\Sigma_1}) = W^* \cap \Big(\bigcap_{\rho\in\Sigma_1} \langle\rho\rangle^\perp\Big) = W^* \cap \langle\Sigma\rangle^\perp.$$

Hence, 4.1.2 implies that c_{N,R,Σ_1} is a monomorphism if and only if Σ is full.

Next, we construct under the condition that Σ is full a canonical morphism from the upper standard sequence $\mathbb{T}_{W,\Sigma}$ to the lower standard sequence $\mathbb{S}_{N,R,\Sigma_1}$, resulting in the so-called standard diagram of Σ.

(4.2.4) By 4.1.2 we have $\operatorname{Ker}(h_{W,\Sigma}) \subseteq \operatorname{Ker}(d'_{N,R,\Sigma})$, and therefore $d'_{N,R,\Sigma}$ induces a morphism $d''_{N,R,\Sigma} : P_{W,\Sigma} \to R^{\Sigma_1}$ in $\mathsf{Mod}(R)$ with $d''_{N,R,\Sigma} \circ h_{W,\Sigma} = d'_{N,R,\Sigma}$. Moreover, $d''_{N,R,\Sigma}$ induces by restriction a morphism

$$d_{N,R,\Sigma} : \overline{P}_{W,\Sigma} \to R^{\Sigma_1}$$

in $\mathsf{Mod}(R)$, and it holds

$$d_{N,R,\Sigma} \circ g_{W,\Sigma} \circ h_{W,\Sigma} \circ \Delta_{W,\Sigma} = c_{N,R,\Sigma_1}.$$

Furthermore, $d_{N,R,\Sigma}$ is a monomorphism. Indeed, let $(m_\sigma)_{\sigma\in\Sigma}$ be a family in W^* with $p := (m_\sigma + \sigma^\vee)_{\sigma\in\Sigma} \in \operatorname{Ker}(d_{N,R,\Sigma})$. Then, we have $m_\rho \in \rho^\perp$ for every $\rho \in \Sigma_1$. Moreover, for every $\sigma \in \Sigma$ and every $\rho \in \sigma_1$ it holds $m_\sigma - m_\rho \in \rho^\perp$, hence $m_\sigma \in \rho^\perp$, therefore $m_\sigma \in \bigcap_{\rho\in\sigma_1} \rho^\perp = \sigma^\perp$ and thus $p = 0$ on use of 4.2.1.

It is clear that $d''_{N,\mathbb{R},\Sigma}$ and $d_{N,\mathbb{R},\Sigma}$ are rational with respect to $P_{W,\Sigma}$ or $\overline{P}_{W,\Sigma}$, respectively, and R^{Σ_1}.

If $R = \mathbb{Z}$ and no confusion about N can arise we write d''_Σ and d_Σ instead of $d''_{N,\mathbb{Z},\Sigma}$ and $d_{N,\mathbb{Z},\Sigma}$.

(4.2.5) Suppose that Σ is full. Then, c_{N,R,Σ_1} is a monomorphism by 4.2.4, and moreover we have an isomorphism $f_{W,\Sigma} : K_{W,\Sigma} \xrightarrow{\cong} W^*$ with

$$c_{N,R,\Sigma_1} \circ f_{W,\Sigma} = d_{N,R,\Sigma} \circ g_{W,\Sigma}.$$

Thus, $\mathrm{Pic}_W(\Sigma)$ being the cokernel of $g_{W,\Sigma}$, there exists a unique morphism

$$b_{N,R,\Sigma} : \mathrm{Pic}_W(\Sigma) \to A_{N,R,\Sigma_1}$$

in $\mathsf{Mod}(R)$ with $b_{N,R,\Sigma} \circ e_{W,\Sigma} = a_{N,R,\Sigma_1} \circ d_{N,R,\Sigma}$. Hence, we have in $\mathsf{Mod}(R)$ the commutative diagram with exact rows

$$\begin{tikzcd}
\mathbb{T}_{W,\Sigma} : & 0 \arrow[r] & K_{W,\Sigma} \arrow[r, "g_{W,\Sigma}"] \arrow[d, "f_{W,\Sigma}"', "\cong"] & \overline{P}_{W,\Sigma} \arrow[r, "e_{W,\Sigma}"] \arrow[d, tail, "d_{N,R,\Sigma}"] & \mathrm{Pic}_W(\Sigma) \arrow[r] \arrow[d, "b_{N,R,\Sigma}"] & 0 \\
\mathbb{S}_{N,R,\Sigma_1} : & 0 \arrow[r] & W^* \arrow[r, "c_{N,R,\Sigma}"] & R^{\Sigma_1} \arrow[r, "a_{N,R,\Sigma}"] & A_{N,R,\Sigma_1} \arrow[r] & 0
\end{tikzcd}$$

denoted by $\mathbb{D}_{N,R,\Sigma}$ and called *the standard diagram of Σ (over R with respect to N)*.

The Snake Lemma implies that $b_{N,R,\Sigma} : \mathrm{Pic}_W(\Sigma) \to A_{N,R,\Sigma_1}$ is a monomorphism. By means of this we identify $\mathrm{Pic}_W(\Sigma)$ with its image in A_{N,R,Σ_1} and hence consider it as a sub-R-module of A_{N,R,Σ_1}.

Clearly, the diagrams $\mathbb{D}_{N,R,\Sigma} \otimes_R \mathbb{R}$ and $\mathbb{D}_{N,\mathbb{R},\Sigma}$ in $\mathsf{Mod}(\mathbb{R})$ are canonically isomorphic and hence identified.

If $R = \mathbb{Z}$ and no confusion about N can arise we write b_Σ and $\mathbb{D}_\Sigma$ instead of $b_{N,\mathbb{Z},\Sigma}$ and $\mathbb{D}_{N,\mathbb{Z},\Sigma}$.

A first application of the lower standard sequence is the following combinatorial characterisation of skeletal completeness of Σ.

(4.2.6) Proposition *Suppose that Σ is full. Then, Σ is skeletal complete if and only if $\mathrm{Im}(c_{N,R,\Sigma_1}) \cap R^{\Sigma_1}_{\geq 0} = 0$.*

Proof. It holds $\mathrm{Im}(c_{N,R,\Sigma_1}) \cap R^{\Sigma_1}_{\geq 0} = 0$ if and only if $W^* \cap (\bigcap_{\rho \in \Sigma_1} \rho^\vee)$ is contained in $\mathrm{Ker}(c_{N,R,\Sigma_1})$. Since Σ is full, it follows from 4.2.4 that this holds if and only if $\bigcap_{\rho \in \Sigma_1} \rho^\vee = 0$, and 1.2.8 implies that this is equivalent to Σ being skeletal complete. □

(4.2.7) If $\mathrm{Im}(c_{N,R,\Sigma_1}) \cap R^{\Sigma_1}_{\geq 0} = 0$, then Σ is not necessarily full; hence we cannot omit the fullness hypothesis in 4.2.6. Indeed, the 1-dimensional $\mathbb{Z}^2$-fan $\Sigma = \{0, \mathrm{cone}((1,0)), \mathrm{cone}((0,1))\}$ in $\mathbb{R}^2$ is a counterexample, for we have

$$c_{\mathbb{Z}^2,\mathbb{Z},\Sigma_1} : \mathbb{Z}^2 \to \mathbb{Z}^2,\ (x,y) \mapsto (x, -x)$$

and hence $\mathrm{Im}(c_{\mathbb{Z}^2,\mathbb{Z},\Sigma_1}) \cap \mathbb{N}_0^2 = 0$.

Finally, we prove the characterisations of regularity and simpliciality of a fan in terms of its Picard group as promised in the preceding section.

(4.2.8) Theorem *Suppose that Σ is full. Then, Σ is W-regular if and only if $\mathrm{Pic}_W(\Sigma) = A_{N,R,\Sigma_1}$.*

Proof. Applying the Snake Lemma to the diagram $\mathbb{D}_{N,R,\Sigma}$ we see that $\mathrm{Pic}_W(\Sigma) = A_{N,R,\Sigma}$ is equivalent to $d_{N,R,\Sigma} : \overline{P}_{W,\Sigma} \rightarrowtail R^{\Sigma_1}$ being an epimorphism. This holds if and only if for every $(r_\rho)_{\rho \in \Sigma_1} \in R^{\Sigma_1}$ there exists a virtual polytope $(m_\sigma + \sigma^\vee)_{\sigma \in \Sigma} \in \overline{P}_{W,\Sigma}$ with $\rho_N(m_\rho) = r_\rho$ for every $\rho \in \Sigma_1$.

Since $\rho_N : W^* \to R$ is surjective for every $\rho \in \Sigma_1$ by 1.4.34 and 1.4.31 b), the above condition is equivalent to condition (ii) in 4.1.14 and therefore to Σ being W-regular. $\square$

(4.2.9) Corollary *Suppose that Σ is full. Then, Σ is simplicial if and only if $\mathrm{Pic}_W(\Sigma)$ has finite index in A_{N,R,Σ_1}.*

Proof. From 2.2.13 we know that Σ is simplicial if and only if it is V-regular, and by 4.2.8 this holds if and only if $A_{N,\mathbb{R},\Sigma_1}/\operatorname{Pic}_V(\Sigma) = 0$. But this is equivalent to $(A_{N,R,\Sigma_1}/\operatorname{Pic}_W(\Sigma)) \otimes_R \mathbb{R} = 0$, and hence to $\operatorname{Pic}_W(\Sigma)$ having finite index in A_{N,R,Σ_1}. $\square$

4.3. Projective systems of monoids defined by fans

Let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, let N be a $\mathbb{Z}$-structure on V, and let $M := N^*$.

We end this chapter on cones and fans by making some connections to the theory from Chapter I. More precisely, we show how an N-fan gives rise to an openly immersive projective system of monoids. We start by showing how N-polycones define certain monoids.

(4.3.1) Proposition *Let A be an N-polycone in V. Then, $A^\vee \cap M$ is a torsionfree, cancellable, finitely generated submonoid of M.*

Proof. Since $A^\vee$ is an M-polycone in V^* by 1.4.6, the set $A^\vee \cap M$ is a submonoid of the free group M and in particular torsionfree and cancellable. Moreover, there is a finite subset $X \subseteq A \cap M$ with $A^\vee = \operatorname{cone}(X)$. We set $Y := \{\sum_{x \in X} r_x x \in V \mid (r_x)_{x \in X} \in [0,1]^X\}$, and we show that $Y \cap M$ is a finite generating set of the monoid $A^\vee \cap M$. Clearly, it is contained in $A^\vee \cap M$. Conversely, denote by L the submonoid of $A^\vee \cap M$ generated by $Y \cap M$, and let $z \in A^\vee \cap M$. Then, there is a family $(r_x)_{x \in X}$ in $\mathbb{R}_{\geq 0}$ with $z = \sum_{x \in X} r_x x$, and for every $x \in X$ there are $r'_x \in \mathbb{N}_0$ and $r''_x \in [0,1[$ with $r_x = r'_x + r''_x$. From this we get $z = \sum_{x \in X} r'_x x + \sum_{x \in X} r''_x x$, and as $\sum_{x \in X} r'_x x \in L \subseteq M$ it follows $\sum_{x \in X} r''_x x \in M$, hence $\sum_{x \in X} r''_x x \in L$ and therefore $z \in L$. Thus, $Y \cap M$ generates the monoid $A^\vee \cap M$.

It is readily checked that $Y = \operatorname{conv}(X \cup \{0\})$, and then 1.2.16 implies that Y is compact. Now, we choose an M-rational basis E of V^*, and for $e \in E$ we denote by $p_e : V \to \mathbb{R}$ the canonical projection which is rational with respect to M and $\mathbb{Z}$. Then, for every $e \in E$ it follows that $p_e(Y \cap M) \subseteq \mathbb{Z}$ is compact and hence finite, since $\mathbb{Z}$ is discrete. But as there is a bijection $Y \cap M \cong \prod_{e \in E} p_e(Y \cap M)$ we see that $Y \cap Ms$ is finite, too, and thus the claim is proven. $\square$

(4.3.2) Proposition *Let A be a sharp N-polycone in V. Then, $A^\vee \cap M$ is finite if and only if $n = 0$.*

Proof. Suppose that $A^\vee \cap M$ is finite. As it is torsionfree and cancellable by 4.3.1 it follows first that the group $\operatorname{Diff}(A^\vee \cap M)$ is finite and torsionfree, hence equal to 0, and therefore $A^\vee \cap M = 0$. But then it holds $A^\vee = \operatorname{cone}(A^\vee \cap M) = 0$ and hence $A = V$ by 1.2.8, and thus sharpness of A implies the claim. □

The above construction of a monoid from a given N-polycone is compatible with facial relations, as we will see now.

(4.3.3) Proposition *Let A, B and C be N-polycones in V with $B \preccurlyeq A$ and $C \preccurlyeq A$. Then, there exists $u \in A^\vee \cap M$ such that $B = A \cap u^\perp$, that $B^\vee \cap M = (A^\vee \cap M) \oplus \mathbb{N}_0(-u)$, that the canonical injection $A^\vee \cap M \hookrightarrow B^\vee \cap M$ equals the canonical morphism $\varepsilon_u : A^\vee \cap M \to A^\vee \cap M - \mathbb{N}_0 u$, and that it holds $C \preccurlyeq B$ if and only if $u \in C^\perp$.*

Proof. If $B = A$, then $u = 0$ fulfils the claim. So, we suppose that $B \prec A$. By 1.4.3 there exists $u \in A^\vee \cap M$ such that $B = A \cap u^\perp$. Let X be a finite generating set of A. By 4.3.1 there is a finite generating set E of the monoid $B^\vee \cap M$, and moreover there exists $m \in \mathbb{N}_0$ such that for every $x \in X$ with $u(x) \neq 0$ and every $e \in E$ it holds $m \geq -\frac{e(x)}{u(x)}$. Now, let $e \in E$. Clearly, we have $e + mu \in M$. For $x \in X$ with $u(x) = 0$ it holds $x \in B$ and hence $(e + mu)(x) \geq 0$, and for $x \in X$ with $u(x) \neq 0$ it holds $(e + mu)(x) \geq 0$ by the choice of m. Therefore, we have $e + mu \in A^\vee \cap M$ and hence $e = (e+mu) - mu \in A^\vee \cap M - \mathbb{N}_0 u$. Thus, it follows $B^\vee \cap M \subseteq A^\vee \cap M - \mathbb{N}_0 u$. Conversely, from 1.4.6 it follows

$$A^\vee \cap M - \mathbb{N}_0 u \subseteq (A^\vee + \langle u \rangle) \cap M = (A \cap u^\perp)^\vee \cap M = B^\vee \cap M.$$

If the sum $A^\vee \cap M + \mathbb{N}_0(-u)$ is not direct, then there is an $m \in \mathbb{N}$ with $-mu \in A^\vee$, yielding $-u \in A^\vee$ and hence $u \in A^\vee \cap (-A)^\vee = A^\perp$. But this implies $A \subseteq A \cap u^\perp = B$ and therefore the contradiction $B = A$.

Finally, $C \preccurlyeq B$ is equivalent to $C \subseteq u^\perp$ by 1.3.1, and hence to $u \in C^\perp$ as claimed. □

(4.3.4) Corollary *Let A and B be N-polycones in V such that $A \cap B \in \operatorname{face}(A) \cap \operatorname{face}(B)$. Then, it holds $(A \cap B)^\vee \cap M = (A^\vee \cap M) \cup (B^\vee \cap M)$.*

Proof. If $A = B$ this is clear. So, let $A \neq B$. Then, by 1.4.14 there is a $u \in A^\vee \cap M$ with $(-u) \in B^\vee \cap M$ such that $A \cap B = A \cap u^\perp$. Then, 4.3.3 implies $(A \cap B)^\vee \cap M = A^\vee \cap M + (-\mathbb{N}_0 u) \subseteq A^\vee \cap M + B^\vee \cap M$. The reverse inclusion follows immediately from 1.2.8. □

Finally, we apply the above to an N-fan Σ to get an openly immersive projective system of certain monoids. In Chapter IV this will be fed to the machinery from Chapter I and thus yield toric schemes.

(4.3.5) Let Σ be an N-fan in V. Keep in mind that we consider Σ as an ordered set by means of the ordering $\preccurlyeq$. Then, Σ is a finite lower semilattice, and for $\sigma, \tau \in \Sigma$ it holds $\inf(\sigma, \tau) = \sigma \cap \tau$. Moreover, if Σ is nonempty, then the zero cone 0 is its smallest element (see 2.1.3).

If σ is a cone in Σ, then $\sigma^\vee$ is an M-polycone in V^* by 1.4.6, and hence $\sigma^\vee_M := \sigma^\vee \cap M$ is a torsionfree, cancellable and finitely generated submonoid of the group M by 4.3.1. Moreover, it holds $\sigma^\vee = \operatorname{cone}(\sigma^\vee_M)$.

If τ is a further cone in Σ with $\tau \preccurlyeq \sigma$, then it holds $\sigma^\vee \subseteq \tau^\vee$ by 1.2.8, and hence $\sigma^\vee_M \subseteq \tau^\vee_M$ is a submonoid. Conversely, if τ is a cone in Σ with $\sigma^\vee_M \subseteq \tau^\vee_M$, then we have $\sigma^\vee \subseteq \tau^\vee$ by the above, hence $\tau \subseteq \sigma$ by 1.2.8 and therefore $\tau \preccurlyeq \sigma$ by 2.1.3. Thus, τ is a face of σ if and only if $\sigma^\vee_M$ is a submonoid of $\tau^\vee_M$.

The family $(\sigma^\vee_M)_{\sigma \in \Sigma}$ defines a projective system of submonoids of the group M over Σ, and by abuse of language we denote this by $\Sigma^\vee_M$.

(4.3.6) Corollary *Let Σ be an N-fan in V. Then, the projective system $\Sigma^\vee_M$ in* Mon *is openly immersive.*

Proof. Clear from 4.3.3 and I.1.4.15. $\square$

CHAPTER III

Graduations

In this chapter we treat graduations of rings and modules in full generality with emphasis on functoriality. For the sake of completeness we include also some well-known results that can be found for example in the comprehensive books [8] and [19] by Năstăsescu and van Oystaeyen.

In Section 1 we start by introducing quasigraduations; the reason for this is to treat graded rings and modules at the same time as graded sets. The latter are used to formulate a graded analogue to free modules. In the context of quasigraduations we introduce two important classes of functors, coarsenings and extensions, that allow to change the set of degrees. Then, graded rings and modules are defined as quasigraded rings and modules that are subject to some algebraic conditions. Moreover, we construct adjoints to coarsening and extension functors for rings and modules and hence get refinement and restriction functors.

Section 2 is devoted to the structure of categories of graded modules. We start by proving that these categories are Abelian and moreover fulfil Grothendieck's axioms AB5 and AB4*, and we investigate the exactness properties of the functors introduced in Section 1. Furthermore, we consider free graded modules, graded Hom functors, and graded projective and injective modules, and we end the section by studying graded rings and modules of fractions. In dealing with graded modules, and especially in this section, two of the main questions are the following:

Given an epimorphism $\psi : G \twoheadrightarrow H$ of groups and a G-graded ring R, which properties of graded modules are respected or reflected by the ψ-coarsening functor $\bullet_{[\psi]} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^H(R_{[\psi]})$?

Given a monomorphism $\varphi : F \rightarrowtail G$ of groups and a G-graded ring R, which properties of graded modules are respected or reflected by the φ-restriction functor $\bullet_{(\varphi)} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^F(R_{(\varphi)})$?

As special cases with $H = 0$ or $F = 0$ respectively these questions treat also the forgetful functors $\mathsf{GrMod}^G(R) \to \mathsf{Mod}(R)$ and the functors of taking components of degree 0 from $\mathsf{GrMod}^G(R)$ to $\mathsf{Mod}(R_0)$.

In Section 3 we collect different results on graded rings and modules that will be used later on. The first two are short notes on strongly graded rings and on saturation, respectively. In the third we treat Noetherianity of graded rings and modules, and we prove graded versions of Hilbert's Basissatz and the Artin-Rees Lemma. Finally, in preparation for the next section on local cohomology we introduce graded versions of torsion functors.

In the last section we study graded homological algebra. After some basics we treat graded Ext functors, and we use them to introduce graded local cohomology functors and graded higher ideal transformation functors. Also here, one of the main questions is whether these functors commute with coarsening functors. Furthermore, we generalise a lot of basic properties of the ungraded or the $\mathbb{N}_0$-graded versions of these functors, as found in the treatise [2] by Brodmann and Sharp, to arbitrary graduations. Finally, we introduce graded Čech cohomology, investigate its behaviour under coarsening and give conditions under which graded local cohomology and graded Čech cohomology are canonically isomorphic.

1. Quasigraduations and graduations

1.1. Quasigraduations

Let C *and* D *be categories, let* $T : \mathsf{C} \to \mathsf{D}$ *be a functor, and assume that* D *has coproducts with monomorphic canonical injections.*

If we take, for example, the usual definition of graded ring, strip it from all additional conditions and formulate what remains in the language of categories, we end up with the notion of quasigraduation as given below (where G is the underlying set of the group of degrees, and T is the forgetful functor from the category Ann of rings to the category Ens of sets).

(1.1.1) Let G be a set. For $A \in \mathrm{Ob}(\mathsf{C})$, a (G,T)-*quasigraduation on* A is a family $(A_g)_{g\in G}$ in $\mathrm{Ob}(\mathsf{D})$ such that

$$T(A) = \coprod_{g\in G} A_g.$$

A (G,T)-*quasigraded object of* C is a pair $(A, (A_g)_{g\in G})$ consisting of an object A of C and a (G,T)-quasigraduation $(A_g)_{g\in G}$ on A. If $(A, (A_g)_{g\in G})$ and $(B, (B_g)_{g\in G})$ are (G,T)-quasigraded objects of C, then a *morphism of* (G,T)*-quasigraded objects of* C *from* $(A, (A_g)_{g\in G})$ *to* $(B, (B_g)_{g\in G})$ is a morphism $u : A \to B$ in C such that there is a family $(u_g)_{g\in G}$ of morphisms in D with $u_g \in \mathrm{Hom}_{\mathsf{D}}(A_g, B_g)$ for every $g \in G$ such that $T(u) = \coprod_{g\in G} u_g$; as coproducts in D have monomorphic canonical injections by hypothesis, this family $(u_g)_{g\in G}$ is unique. The (G,T)-quasigraded objects of C and the morphisms of such form a category, denoted by $\mathsf{QGrC}^{G,T}$.

If no confusion about T can arise, by abuse of language we will just speak of G-quasigraduations, G-quasigraded objects, and morphisms of G-quasigraded objects, and we will write QGrC^{G} instead of $\mathsf{QGrC}^{G,T}$. Moreover, we will denote a G-quasigraded object $(A, (A_g)_{g\in G})$ in C just by A.

If G is, instead of a set, an object of a category that is furnished with a forgetful functor to Ens, then by abuse of language we will in the above notations use G instead of some symbol for the set underlying G.

(1.1.2) Example Let G be a set, and let $T = \mathrm{Id}_{\mathsf{Ens}} : \mathsf{Ens} \to \mathsf{Ens}$. If $a : A \to G$ is a set over G, then $(A, (a^{-1}(g))_{g\in G})$ is a G-quasigraded set. If $a : A \to G$ and

$b : B \to G$ are sets over G and $u : A \to B$ is a morphism in $\mathsf{Ens}_{/G}$, then u is a morphism of G-quasigraded sets from $(A, (a^{-1}(g))_{g \in G})$ to $(B, (b^{-1}(g))_{g \in G})$. This gives rise to a faithful functor

$$\mathsf{Ens}_{/G} \to \mathsf{QGrEns}^{G,\mathrm{Id}_{\mathsf{Ens}}}.$$

Now we introduce some standard functors on categories of quasigraded objects, starting with the functor that forgets the quasigraduation and with the functor of taking components of some given degree, and going on with coarsening and extension functors. (It might be tempting to define "restriction functors", but this is easily seen to not make sense in this general setting.)

(1.1.3) If G is a set, then there is a forgetful functor $\mathsf{QGrC}^G \to \mathsf{C}$, mapping a G-quasigraded object in C onto its underlying object in C. If $\mathrm{Card}(G) = 1$, then this is an isomorphism of categories by means of which we identify these two categories.

(1.1.4) Let G be a set, and let $g \in G$. Then, there is a functor

$$\bullet_g : \mathsf{QGrC}^G \to \mathsf{D},$$

mapping a G-quasigraded object $(A, (A_g)_{g \in G})$ in C onto A_g, called *the component of degree g of A,* and mapping a morphism $u : A \to B$ in QGrC^G onto the uniquely determined morphism $u_g : A_g \to B_g$ in D, called *the component of degree g of u.* In case $\mathrm{Card}(G) = 1$ this functor coincides with T.

(1.1.5) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ens. If A is a (G, T)-quasigraded object in C, we define an (H, T)-quasigraded object $A_{[\psi]}$ in C, its underlying object of C being A, and its (H, T)-quasigraduation being $(\coprod_{g \in \psi^{-1}(h)} T(A_g))_{h \in H}$. If $u : A \to B$ is a morphism in QGrC^G, then it is also a morphism in QGrC^H from $A_{[\psi]}$ to $B_{[\psi]}$; we denoted it by $u_{[\psi]}$. This gives rise to a faithful functor

$$\bullet_{[\psi]} : \mathsf{QGrC}^{G,T} \to \mathsf{QGrC}^{H,T},$$

called *the ψ-coarsening.*

If $\mathrm{Card}(H) = 1$, then this coincides with the forgetful functor from $\mathsf{QGrC}^{G,T}$ to C (see 1.1.3). If $\psi' : H \twoheadrightarrow H'$ is a further epimorphism in Ens, then it holds

$$\bullet_{[\psi' \circ \psi]} = (\bullet_{[\psi]})_{[\psi']}.$$

(1.1.6) Let $\varphi : F \rightarrowtail G$ be a monomorphism in Ens, and denote by I the initial object of D. If A is an (F, T)-quasigraded object in C, then we define a (G, T)-quasigraded object $A^{(\varphi)}$ in C, its underlying object in C being A, and its (G, T)-quasigraduation being given by $(A^{(\varphi)})_g = A_f$ for $g \in G$ such that there is a (necessarily unique) $f \in F$ with $\varphi(f) = g$, and by $(A^{(\varphi)})_g = I$ for $g \in G \setminus \varphi(F)$. If $u : A \to B$ is a morphism in QGrC^F, then it is also a morphism in QGrC^G from $A^{(\varphi)}$ to $B^{(\varphi)}$; we denote it by $u^{(\varphi)}$. This gives rise to a faithful functor

$$\bullet^{(\varphi)} : \mathsf{QGrC}^{F,T} \to \mathsf{QGrC}^{G,T},$$

called *the φ-extension.*

If $\varphi' : F' \rightarrowtail F$ is a further monomorphism in Ens, then it holds

$$\bullet^{(\varphi\circ\varphi')} = (\bullet^{(\varphi')})^{(\varphi)}.$$

In case C consists of groups with some additional structure, we can define degree map and degree support.

(1.1.7) Let G be a set, and suppose that $\mathsf{D} = \mathsf{Ab}$, that C is a category of groups with some additional structure, and that T is the forgetful functor. Moreover, let A be a G-quasigraded object of C. We set $A^{\mathrm{hom}} := \bigcup_{g\in G} A_g$. Then, there is a unique map

$$\deg : A^{\mathrm{hom}} \setminus \{0\} \to G,$$

called *the degree map on A,* such that for every $a \in A^{\mathrm{hom}} \setminus \{0\}$ it holds $a \in A_{\deg(a)}$. Furthermore, the set

$$\mathrm{degsupp}(A) := \{g \in G \mid A_g \neq 0\}$$

is called *the degree support of A.*

Now, let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ens. Then, it holds $A^{\mathrm{hom}} \subseteq (A_{[\psi]})^{\mathrm{hom}}$, and the restriction of the degree map on $A_{[\psi]}$ to $A^{\mathrm{hom}} \setminus \{0\}$ is the degree map on A. Moreover, it is easily seen that

$$\mathrm{degsupp}(A_{[\psi]}) = \psi(\mathrm{degsupp}(A)).$$

Finally, in case the set G of degrees carries a structure of group we can define the shift functors.

(1.1.8) Let G be a group, and let $g \in G$. For a (G,T)-quasigraded object A of C, we define a (G,T)-quasigraded object $A(g)$ of C, its underlying object of C being A, and its (G,T)-quasigraduation being $(A_{h+g})_{h\in G}$. If $u : A \to B$ is a morphism in QGrC^G, then it is also a morphism in QGrC^G from $A(g)$ to $B(g)$; we denote it by $u(g)$. This gives rise to a functor

$$\bullet(g) : \mathsf{QGrC}^{G,T} \to \mathsf{QGrC}^{G,T},$$

called *the g-shift.* This is an isomorphism of categories, and its inverse is $\bullet(-g)$. Moreover, the diagram of categories

$$\begin{tikzcd}
\mathsf{QGrC}^{G,T} \arrow[rr, "\bullet(g)"] \arrow[dr, "\bullet_g"'] & & \mathsf{QGrC}^{G,T} \arrow[dl, "\bullet_0"] \\
& \mathsf{D} &
\end{tikzcd}$$

commutes.

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then the diagram of categories

$$\begin{tikzcd} \mathsf{QGrC}^{G,T} \arrow[r, "\bullet(g)"] \arrow[d, "\bullet_{[\psi]}"'] & \mathsf{QGrC}^{G,T} \arrow[d, "\bullet_{[\psi]}"] \\ \mathsf{QGrC}^{H,T} \arrow[r, "\bullet(\psi(g))"] & \mathsf{QGrC}^{H,T} \end{tikzcd}$$

commutes.

(1.1.9) Concerning set theory, the only point to consider is 1.1.1. If C is a $\mathcal{U}$-category, then so is $\mathsf{QGrC}^{G,T}$, for the forgetful functor $\mathsf{QGrC}^{G,T} \to \mathsf{C}$ is faithful.

Moreover, if C, D and G are $\mathcal{U}$-small, then so is $\mathsf{QGrC}^{G,T}$. Indeed, by the above it suffices to show that the set of objects of $\mathsf{QGrC}^{G,T}$ is $\mathcal{U}$-small, and for this it suffices to show that the set of objects in $\mathsf{QGrC}^{G,T}$ with underlying object in C a given $A \in \mathrm{Ob}(\mathsf{C})$ is $\mathcal{U}$-small. But if $A \in \mathrm{Ob}(\mathsf{C})$, then the set of structures of (G,T)-quasigraduations on A is a subset of $\mathrm{Ob}(\mathsf{D})^G$ and hence $\mathcal{U}$-small by [1, I.11.1 Proposition 6]. The same argument shows that if C, D and G are elements of $\mathcal{U}$, then so is $\mathsf{QGrC}^{G,T}$.

1.2. Graded rings and modules

Let G be a group.

In this section we impose some conditions on quasigraded rings and quasigraded modules in order to get the usual notions of graded rings and graded modules.

(1.2.1) A *G-graded ring* is a (G,T)-quasigraded ring $(R,(R_g)_{g\in G})$, where $T : \mathsf{Ann} \to \mathsf{Ab}$ denotes the forgetful functor, such that $R_g R_h \subseteq R_{g+h}$ for all $g,h \in G$. We denote by GrAnn^G the full subcategory of QGrAnn^G whose objects are the G-graded rings.

(1.2.2) The functor $\bullet_0 : \mathsf{QGrAnn}^G \to \mathsf{Ab}$ induces a functor

$$\bullet_0 : \mathsf{GrAnn}^G \to \mathsf{Ann}$$

such that the diagram of categories

$$\begin{tikzcd} \mathsf{QGrAnn}^G \arrow[r, "\bullet_0"] & \mathsf{Ab} \\ \mathsf{GrAnn}^G \arrow[r, "\bullet_0"] \arrow[u] & \mathsf{Ann}, \arrow[u] \end{tikzcd}$$

where the unmarked functors are the canonical injection and the forgetful functor, respectively, commutes. Furthermore, for every $g \in G$ the functor $\bullet_g : \mathsf{QGrAnn}^G \to \mathsf{Ab}$ induces by restriction a functor

$$\bullet_g : \mathsf{GrAnn}^G \to \mathsf{Ab}.$$

If R is a G-graded ring and $g \in G$, then R_g is canonically furnished with a structure of R_0-module.

(1.2.3) Let R be a G-graded ring. A *G-graded R-module* is a (G,T)-quasigraded R-module $(M,(M_g)_{g\in G})$, where $T:\mathsf{Mod}(R)\rightarrow\mathsf{Ab}$ denotes the forgetful functor, such that $R_gM_h\subseteq M_{g+h}$ for all $g,h\in G$. We denote by $\mathsf{GrMod}^G(R)$ the full subcategory of $\mathsf{QGrMod}(R)^G$ whose objects are the G-graded R-modules.

(1.2.4) Let R be a G-graded ring, and let $g\in G$. Then, the functor $\bullet_g:\mathsf{QGrMod}(R)^G\rightarrow\mathsf{Ab}$ induces a functor

$$\bullet_g:\mathsf{GrMod}^G(R)\rightarrow\mathsf{Mod}(R_0)$$

such that the diagram of categories

$$\begin{tikzcd}
\mathsf{QGrMod}(R)^G \arrow[r, "\bullet_g"] & \mathsf{Ab} \\
\mathsf{GrMod}^G(R) \arrow[u] \arrow[r, "\bullet_g"] & \mathsf{Mod}(R_0), \arrow[u]
\end{tikzcd}$$

where the unmarked functors are the canonical injection and the forgetful functor, repsectively, commutes.

Moreover, the functor $\bullet(g):\mathsf{QGrMod}(R)^G\rightarrow\mathsf{QGrMod}(R)^G$ induces by restriction and coastriction a functor

$$\bullet(g):\mathsf{GrMod}^G(R)\rightarrow\mathsf{GrMod}^G(R).$$

This is again an isomorphism of categories, and its inverse is $\bullet(-g)$.

(1.2.5) Let R be a G-graded ring. A *G-graded R-algebra* is a G-graded ring under R, and a *morphism of G-graded R-algebras* is a morphism of G-graded rings under R. We denote by $\mathsf{GrAlg}^G(R)$ the category $(\mathsf{GrAnn}^G)^{/R}$ of G-graded R-algebras.

(1.2.6) Concerning set theory, it is clear from 1.1.9 that GrAnn^G, $\mathsf{GrMod}^G(R)$ and $\mathsf{GrAlg}^G(R)$ are $\mathcal{U}$-categories. Moreover, when dealing with graded rings, or graded modules and graded algebras, respectively, we suppose from now on that the group G of degrees, and the base ring R, respectively, are elements of $\mathcal{U}$. Then, the objects of the categories GrAnn^G, $\mathsf{GrMod}^G(R)$ and $\mathsf{GrAlg}^G(R)$ are elements of $\mathcal{U}$, too.

1.3. Coarsening and refinement

Let $\psi:G\twoheadrightarrow H$ be an epimorphism in Ab*.*

Coarsening functors on quasigraded rings induce coarsening functors on graded rings, and we will now construct right adjoints of these, called refinement functors.

(1.3.1) By restriction and coastriction, ψ-coarsening on QGrAnn^G induces a faithful functor

$$\bullet_{[\psi]}:\mathsf{GrAnn}^G\rightarrow\mathsf{GrAnn}^H.$$

(1.3.2) Proposition *There is a functor*

$$\bullet^{[\psi]} : \mathsf{GrAnn}^H \to \mathsf{GrAnn}^G$$

that is right adjoint to

$$\bullet_{[\psi]} : \mathsf{GrAnn}^G \to \mathsf{GrAnn}^H.$$

Proof. For an H-graded ring R, we define a G-graded ring $R^{[\psi]}$ as follows. For $g \in G$ we set $R^{[\psi]}_g := R_{\psi(g)}$, and we take $\bigoplus_{g\in G} R^{[\psi]}_g$ as the additive group underlying $R^{[\psi]}$. Multiplication of R induces maps

$$R^{[\psi]}_g \times R^{[\psi]}_h \to R^{[\psi]}_{g+h}$$

for all $g, h \in G$, and these yield a structure of ring on $\bigoplus_{g\in G} R^{[\psi]}_g$. Finally, the G-graduation of $R^{[\psi]}$ is $(R^{[\psi]}_g)_{g\in G}$. If $u : R \to S$ is a morphism in GrAnn^H, then there is a morphism

$$u^{[\psi]} := \bigoplus_{g\in G} u_{\psi(g)} : R^{[\psi]} \to S^{[\psi]}$$

in GrAnn^G. This gives rise to a functor $\bullet^{[\psi]} : \mathsf{GrAnn}^H \to \mathsf{GrAnn}^G$.

For a G-graded ring R, the canonical injections

$$R_g \to (R_{[\psi]})_{\psi(g)} = ((R_{[\psi]})^{[\psi]})_g$$

for $g \in G$ yield a monomorphism $R \to (R_{[\psi]})^{[\psi]}$ in GrAnn^G. As this is natural in R, we get a monomorphism of functors $\mathrm{Id}_{\mathsf{GrAnn}^G} \to (\bullet_{[\psi]})^{[\psi]}$. On the other hand, Id_{R_g} is a morphism $(R^{[\psi]})_h \to R_g$ in Ab for every $g \in G$ and every $h \in \psi^{-1}(g)$. These morphisms yield a morphism $((R^{[\psi]})_{[\psi]})_g \to R_g$ in Ab for every $g \in G$. From these we get a morphism $(R^{[\psi]})_{[\psi]} \to R$ in GrAnn^G, and as it is natural in R it gives rise to a morphism of functors $(\bullet^{[\psi]})_{[\psi]} \to \mathrm{Id}_{\mathsf{GrAnn}^H}$. Using these morphisms

$$\mathrm{Id}_{\mathsf{GrAnn}^G} \to (\bullet_{[\psi]})^{[\psi]} \text{ and } (\bullet^{[\psi]})_{[\psi]} \to \mathrm{Id}_{\mathsf{GrAnn}^H}$$

it is straightforward to prove the claim. □

(1.3.3) The right adjoint $\bullet^{[\psi]} : \mathsf{GrAnn}^H \to \mathsf{GrAnn}^G$ of ψ-coarsening is called *the ψ-refinement.* If $\psi' : H \twoheadrightarrow H'$ is a further epimorphism in Ab, then it holds

$$(\bullet^{[\psi']})^{[\psi]} = \bullet^{[\psi'\circ\psi]}.$$

Also on graded modules we have coarsening functors, and as above with rings we will now construct refinement functors as right adjoint of coarsening functors. Depending with what ring we start there are two variants of this task.

(1.3.4) Let R be a G-graded ring. Then, $\mathsf{GrMod}^H(R_{[\psi]})$ is a full subcategory of $\mathsf{QGrMod}(R)^H$, and ψ-coarsening on $\mathsf{QGrMod}(R)^G$ induces by restriction and coastriction a faithful functor

$$\bullet_{[\psi]} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^H(R_{[\psi]}).$$

(1.3.5) Proposition *Let R be a G-graded ring. There is a functor*

$$\bullet^{[\psi]} : \mathsf{GrMod}^H(R_{[\psi]}) \to \mathsf{GrMod}^G(R)$$

that is right adjoint to

$$\bullet_{[\psi]} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^H(R_{[\psi]}).$$

Proof. First, let S be an H-graded ring. If M is an H-graded S-module, then we define a G-graded $S^{[\psi]}$-module $M^{[\psi]}$ as follows. For $g \in G$ we set $M_g^{[\psi]} := M_{\psi(g)}$, and we take $\bigoplus_{g\in G} M_g^{[\psi]}$ as the group underlying $M^{[\psi]}$. The structure of S-module of M induces maps

$$S_g^{[\psi]} \times M_h^{[\psi]} \to M_{g+h}^{[\psi]}$$

for all $g, h \in G$. These yield a structure of $S^{[\psi]}$-module on $\bigoplus_{g\in G} M_g^{[\psi]}$. Finally, the G-graduation of $M^{[\psi]}$ is $(M_g^{[\psi]})_{g\in G}$. If $u : M \to N$ is a morphism in $\mathsf{GrMod}^H(S)$, then there is a morphism

$$u^{[\psi]} := \bigoplus_{g\in G} u_{\psi(g)} : M^{[\psi]} \to N^{[\psi]}$$

in $\mathsf{GrMod}^G(S^{[\psi]})$. Clearly, this gives rise to a functor

$$\bullet^{[\psi]} : \mathsf{GrMod}^H(S) \to \mathsf{GrMod}^G(S^{[\psi]}).$$

Now, consider $S = R_{[\psi]}$. Composing

$$\bullet^{[\psi]} : \mathsf{GrMod}^H(R_{[\psi]}) \to \mathsf{GrMod}^G((R_{[\psi]})^{[\psi]})$$

with scalar restriction by means of the canonical monomorphism from R to $(R_{[\psi]})^{[\psi]}$, we get a functor $\mathsf{GrMod}^H(R_{[\psi]}) \to \mathsf{GrMod}^G(R)$ that is again denoted by $\bullet^{[\psi]}$.

Let M be a G-graded R-module. The canonical injections

$$M_g \to (M_{[\psi]})_{\psi(g)} = ((M_{[\psi]})^{[\psi]})_g$$

in Ab for $g \in G$ yield a monomorphism $M \to (M_{[\psi]})^{[\psi]}$ in $\mathsf{GrMod}^G(R)$. As this is clearly natural in M, we get a monomorphism of functors

$$\mathrm{Id}_{\mathsf{GrMod}^G(R)} \to (\bullet_{[\psi]})^{[\psi]}.$$

On the other hand, Id_{M_g} is a morphism $(M^{[\psi]})_h \to M_g$ in Ab for every $g \in G$ and every $h \in \psi^{-1}(g)$. These morphisms yield a morphism

$$((M^{[\psi]})_{[\psi]})_g \to M_g$$

in Ab for every $g \in G$, and from these we get a morphism

$$(M^{[\psi]})_{[\psi]} \to M$$

in $\mathsf{GrMod}^G(R)$. As it is natural in M, we get a morphism of functors $(\bullet^{[\psi]})_{[\psi]} \to \mathrm{Id}_{\mathsf{GrMod}^H(R_{[\psi]})}$. Using these morphisms

$$\mathrm{Id}_{\mathsf{GrMod}^G(R)} \to (\bullet_{[\psi]})^{[\psi]} \text{ and } (\bullet^{[\psi]})_{[\psi]} \to \mathrm{Id}_{\mathsf{GrMod}^H(R_{[\psi]})}$$

it is straightforward to prove the claim. $\square$

(1.3.6) For an H-graded ring S, the functor

$$\bullet^{[\psi]} : \mathsf{GrMod}^H(S) \to \mathsf{GrMod}^G(S^{[\psi]})$$

constructed in the proof of 1.3.5 is called *the ψ-refinement.* For a G-graded ring, the right adjoint $\bullet^{[\psi]} : \mathsf{GrMod}^H(R_{[\psi]}) \to \mathsf{GrMod}^G(R)$ of ψ-coarsening is also called *the ψ-refinement.* If $\psi' : H \twoheadrightarrow H'$ is a further epimorphism in Ab, then it holds

$$(\bullet^{[\psi']})^{[\psi]} = \bullet^{[\psi' \circ \psi]}.$$

1.4. Extension and restriction

Let $\varphi : F \rightarrowtail G$ be a monomorphism in Ab.

Extension functors on quasigraded rings induce extension functors on graded rings, and we will now construct right adjoints of these, called restriction functors.

(1.4.1) By restriction and coastriction, φ-extension on QGrAnn^H induces a faithful functor

$$\bullet^{(\varphi)} : \mathsf{GrAnn}^F \to \mathsf{GrAnn}^G.$$

(1.4.2) Proposition *There is a functor*

$$\bullet_{(\varphi)} : \mathsf{GrAnn}^G \to \mathsf{GrAnn}^F$$

that is right adjoint to

$$\bullet^{(\varphi)} : \mathsf{GrAnn}^F \to \mathsf{GrAnn}^G.$$

Proof. If R is a G-graded ring, then we define an F-graded ring $R_{(\varphi)}$, its underlying ring being the subring $\bigoplus_{f \in F} R_{\varphi(f)}$ of R and its F-graduation being $(R_{(\varphi(f))})_{f \in F}$. If $u : R \to S$ is a morphism in GrAnn^G, then there is a morphism $u_{(\varphi)} := \bigoplus_{f \in F} u_{\varphi(f)} : R_{(\varphi)} \to S_{(\varphi)}$ in GrAnn^F. This gives rise to a functor

$$\bullet_{(\varphi)} : \mathsf{GrAnn}^G \to \mathsf{GrAnn}^F.$$

It holds $(\bullet^{(\varphi)})_{(\varphi)} = \mathrm{Id}_{\mathsf{GrAnn}^F}$, and $(\bullet_{(\varphi)})^{(\varphi)}$ is a subfunctor of $\mathrm{Id}_{\mathsf{GrAnn}^G}$. Using the identity morphism $\mathrm{Id}_{\mathsf{GrAnn}^F} \to (\bullet^{(\varphi)})_{(\varphi)}$ and the canonical injection $(\bullet_{(\varphi)})^{(\varphi)} \to \mathrm{Id}_{\mathsf{GrAnn}^G}$ it is straightforward to prove the claim. □

(1.4.3) The right adjoint $\bullet_{(\varphi)} : \mathsf{GrAnn}^G \to \mathsf{GrAnn}^F$ of φ-extension is called *the φ-restriction.* If $\varphi' : F' \rightarrowtail F$ is a further monomorphism in Ab, then it holds

$$(\bullet_{(\varphi)})_{(\varphi')} = \bullet_{(\varphi \circ \varphi')}.$$

Also on graded modules we have extension functors, and as above with rings we will now try to construct restriction functors as right adjoints. Again, there are two variants of the task, and we will find a right adjoint only for one of these variants. However, for the other variant we can construct a restriction functor directly, and we will moreover show in 1.4.8 that this is right adjoint to some other variant of "extension functor".

(1.4.4) Let S be an F-graded ring. If M is an F-graded S-module, then the G-quasigraded S-module $M^{(\varphi)}$ is furnished canonically with a structure of G-graded $S^{(\varphi)}$-module, and therefore φ-extension on $\mathsf{QGrMod}^F(S)$ induces a faithful functor

$$\bullet^{(\varphi)} : \mathsf{GrMod}^F(S) \to \mathsf{GrMod}^G(S^{(\varphi)}).$$

(1.4.5) Proposition *Let S be an F-graded ring. Then, there is a functor*

$$\bullet_{(\varphi)} : \mathsf{GrMod}^G(S^{(\varphi)}) \to \mathsf{GrMod}^F(S)$$

that is right adjoint to

$$\bullet^{(\varphi)} : \mathsf{GrMod}^F(S) \to \mathsf{GrMod}^G(S^{(\varphi)}).$$

Proof. First, let R be a G-graded ring. If M is a G-graded R-module, then we define an F-graded $R_{(\varphi)}$-module, with underlying $R_{(\varphi)}$-module the sub-$R_{(\varphi)}$-module $\bigoplus_{f\in F} M_{\varphi(f)}$ of M and with F-graduation $(M_{(\varphi(f))})_{f\in F}$. If $u : M \to N$ is a morphism in $\mathsf{GrMod}^G(R)$, then there is a morphism

$$u_{(\varphi)} := \bigoplus_{f\in F} u_{\varphi(f)} : M_{(\varphi)} \to N_{(\varphi)}$$

in $\mathsf{GrMod}^F(R_{(\varphi)})$. This gives rise to a functor

$$\bullet_{(\varphi)} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^F(R_{(\varphi)}).$$

Now, we consider $R = S^{(\varphi)}$. It holds $(\bullet^{(\varphi)})_{(\varphi)} = \mathrm{Id}_{\mathsf{GrMod}^F(S)}$, and $(\bullet_{(\varphi)})^{(\varphi)}$ is a subfunctor of $\mathrm{Id}_{\mathsf{GrMod}^G(R)}$. Then, on use of the identity morphism $\mathrm{Id}_{\mathsf{GrMod}^F(S)} \to (\bullet^{(\varphi)})_{(\varphi)}$ and the canonical injection from $(\bullet_{(\varphi)})^{(\varphi)}$ into $\mathrm{Id}_{\mathsf{GrMod}^G(R^{(\varphi)})}$ it is straightforward to prove the claim. □

(1.4.6) For a G-graded ring R, the functor

$$\bullet_{(\varphi)} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^F(R_{(\varphi)})$$

constructed in the proof of 1.4.5 is called *the φ-restriction.* For a further monomorphism $\varphi' : F' \rightarrowtail F$ in Ab it holds

$$(\bullet_{(\varphi)})_{(\varphi')} = \bullet_{(\varphi\circ\varphi')}.$$

Moreover, for every $f \in F$ it holds

$$\bullet(f)_{(\varphi)} = \bullet_{(\varphi)}(\varphi(f)).$$

(1.4.7) If F is a subgroup of G and φ is the canonical injection, then by abuse of language we denote the functors $\bullet^{(\varphi)}$ and $\bullet_{(\varphi)}$ by $\bullet^{(G)}$ and $\bullet_{(F)}$, respectively, and we call them *the G-extension* and *the F-restriction.* We use the notation $\bullet^{(G)}$ in particular in case $F = 0$.

(1.4.8) Proposition *Let R be a G-graded ring. Then, there is a functor*

$$\mathsf{GrMod}^F(R_{(\varphi)}) \to \mathsf{GrMod}^G(R)$$

that is left adjoint to

$$\bullet_{(\varphi)} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^F(R_{(\varphi)}).$$

Proof. For an F-graded $R_{(\varphi)}$-module M we define a G-graded R-module $T(M)$, its underlying R-module being $R \otimes_{R_{(\varphi)}} M$ and its G-graduation being given by

$$T(M)_g = \bigoplus \{R_h \otimes_{R_0} M_f \mid h \in G \wedge f \in F \wedge h + \varphi(f) = g\}$$

for every $g \in G$. For a morphism $u : M \to N$ in in $\mathsf{GrMod}^F(R_{(\varphi)})$, we have a morphism

$$T(u) := R \otimes_{R_{(\varphi)}} u : T(M) \to T(N)$$

in $\mathsf{GrMod}^G(R)$. This gives rise to a functor

$$T : \mathsf{GrMod}^F(R_{(\varphi)}) \to \mathsf{GrMod}^G(R).$$

There is a canonical isomorphism of functors

$$\mathrm{Id}_{\mathsf{GrMod}^F(R_{(\varphi)})} \xrightarrow{\cong} T(\bullet)_{(\varphi)}.$$

For a G-graded R-module M we have a morphism $T(M_{(\varphi)}) \to M$ with $r \otimes x \mapsto rx$ in $\mathsf{GrMod}^G(R)$. Since this is natural in M, we get a morphism of functors $T(\bullet_{(\varphi)}) \to \mathrm{Id}_{\mathsf{GrMod}^G(R)}$. Using this and the above isomorphism it is straightforward to prove the claim. $\square$

(1.4.9) Let R be a G-graded ring. If no confusion can arise, we will denote the left adjoint of $\bullet_{(\varphi)} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^F(R_{(\varphi)})$ just by

$$R \otimes_{R_{(\varphi)}} \bullet : \mathsf{GrMod}^F(R_{(\varphi)}) \to \mathsf{GrMod}^G(R).$$

(1.4.10) Let $h : R \to S$ be a morphism in GrAnn^F. Then, the diagram of categories

$$\begin{tikzcd}
\mathsf{GrAlg}^F(R) \arrow[r, "\bullet^{(\varphi)}"] \arrow[d, "\bullet\otimes_R S"'] & \mathsf{GrAlg}^G(R^{(\varphi)}) \arrow[r, "\bullet_{(\varphi)}"] \arrow[d, "\bullet\otimes_{R^{(\varphi)}} S^{(\varphi)}"'] & \mathsf{GrAlg}^F(R) \arrow[d, "\bullet\otimes_R S"] \\
\mathsf{GrAlg}^F(S) \arrow[r, "\bullet^{(\varphi)}"] & \mathsf{GrAlg}^G(S^{(\varphi)}) \arrow[r, "\bullet_{(\varphi)}"] & \mathsf{GrAlg}^F(S)
\end{tikzcd}$$

quasicommutes.

(1.4.11) Proposition *Let R be a G-graded ring, let M be a G-graded R-module, and let $N \subseteq M_{(\varphi)}$ be an F-graded sub-$R_{(\varphi)}$-module. Then, it holds $N = \langle N \rangle_R \cap M_{(\varphi)}$.*

Proof. Let $f \in F$, $g \in G$, $x \in N \cap M_{\varphi(f)}$ and $r \in R_g$ be such that $rx \in M_{(\varphi)}$. Then, we have $g + \varphi(f) = \deg(rx) \in \varphi(F)$ and hence $g \in \varphi(F)$, and this implies $rx \in R_{(\varphi)}N \subseteq N$. Therefore it holds $\langle N \rangle_R \cap M_{(\varphi)} \subseteq N$. The other inclusion is obvious. □

(1.4.12) Corollary *Let R be a G-graded ring, let M be a G-graded R-module, let $g \in G$, and let $N \subseteq M_g$ be a sub-R_0-module. Then, it holds $N = \langle N \rangle_R \cap M_g$.*

Proof. Apply 1.4.11 with $F = 0$ to $M(g)$. □

2. Categories of graded modules

2.1. Abelianity of categories of graded modules

Let G be a group, and let R be a G-graded ring.

In this section we prove that categories of graded modules are Abelian, and we investigate exactness properties of some of the standard functors introduced in the above sections.

(2.1.1) Proposition *a) The category* $\mathsf{GrMod}^G(R)$ *is Abelian and fulfils AB5 and AB4*.*

b) The forgetful functor $\mathsf{GrMod}^G(R) \to \mathsf{Mod}(R)$ *commutes with inductive limits and with finite projective limits.*

c) For every $g \in G$, the functor $\bullet_g : \mathsf{GrMod}^G(R) \to \mathsf{Mod}(R_0)$ *commutes with inductive limits and with finite projective limits.*

Proof. Let $V : \mathsf{GrMod}^G(R) \to \mathsf{Mod}(R)$ and $W : \mathsf{Mod}(R) \to \mathsf{Ab}$ denote the forgetful functors.

First, let $u, v : M \to N$ be two parallel morphisms in $\mathsf{GrMod}^G(R)$, and let $i : K \rightarrowtail V(M)$ and $p : V(N) \twoheadrightarrow L$ be the kernel and the cokernel of $(V(u), V(v))$. Then, $(W(K) \cap M_g)_{g\in G}$ is a G-graduation on K, and $((M_g + \operatorname{Im}(W(p)))/\operatorname{Im}(W(p)))_{g\in G}$ is a G-graduation on L. Moreover, if we furnish K and L with these, then i and p are morphisms in $\mathsf{GrMod}^G(R)$, and it is easy to check that they are a kernel and a cokernel respectively of (u, v).

Now, let $(M_i)_{i\in I}$ be a family of G-graded R-modules and let $Q := \bigoplus_{i\in I} V(M_i)$. Then, $(\bigoplus_{i\in I} W(M_i)_g)_{g\in G}$ is a G-graduation on Q, and if we furnish Q with this, then the canonical injection from $V(M_j)$ to Q is a morphism in $\mathsf{GrMod}^G(R)$ for every $j \in I$. This yields a coproduct of $(M_i)_{i\in I}$, as is easy to check. Finally, let P denote the sub-R-module $\bigoplus_{g\in G} \prod_{i\in I} (M_i)_g$ of $\prod_{i\in I} V(M_i)$. Then, $(\prod_{i\in I} (M_i)_g)_{g\in G}$ is a G-graduation on P, and if we furnish P with this, then the restriction to P of the canonical projection from $\prod_{i\in I} V(M_i)$ onto $V(M_j)$ is a morphism in $\mathsf{GrMod}^G(R)$ for every $j \in I$. Again it is easy to check that this yields a product of $(M_i)_{i\in I}$.

Hence, $\mathsf{GrMod}^G(R)$ has kernels and cokernel of pairs of parallel morphisms, and products and coproduct. Therefore it has inductive and projective limits by [1, Proposition I.2.3]. Moreover, we have seen above that V commutes with coproducts, with finite products and with kernels and cokernels of pairs of parallel morphisms. Hence, it commutes with inductive limits and with finite projective limits by [1, I.2.4.2]. Furthermore, for $g \in G$ it is clear from the above that $\bullet_g$ commutes with inductive limits and with finite projective limits.

For G-graded R-modules M and N, the set $\operatorname{Hom}_{\mathsf{GrMod}^G(R)}(M, N)$ is a subgroup of $\operatorname{Hom}_R(M, N)$. As V is faithful and commutes with kernels and cokernels, it follows that $\mathsf{GrMod}^G(R)$ is Abelian. Moreover, as V commutes with inductive limits and is exact and faithful, $\mathsf{GrMod}^G(R)$ fulfils AB5.

Finally, let $(u_i)_{i\in I}$ be a family of epimorphisms in $\mathsf{GrMod}^G(R)$. Then, from the above construction it is seen that

$$W(V(\prod_{i\in I} u_i)) = \bigoplus_{g\in G}\prod_{i\in I}(u_i)_g.$$

As $\bullet_g$ is exact for every $g \in G$, the morphism $(u_i)_g$ in Ab is an epimorphism for every $g \in G$ and every $i \in I$. Since Ab fulfils AB4*, it follows that $W(V(\prod_{i\in I} u_i))$ is an epimorphism in Ab. As $W \circ V$ is faithful, $\prod_{i\in I} u_i$ is an epimorphism. Thus, $\mathsf{GrMod}^G(R)$ fulfils AB4*. □

(2.1.2) Lemma *Let I be a category, let C, D and E be Abelian categories with inductive I-limits, and let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ be additive functors. Moreover, suppose that G is faithful and exact and that G and $G \circ F$ commute with inductive I-limits. Then, F commutes with inductive I-limits.*

Proof. Let $L : I \to \mathsf{C}$ be a functor. As G and $G \circ F$ commute with inductive I-limits, the canonical morphisms

$$\varinjlim_I (G \circ F \circ L) \to G(\varinjlim_I (F \circ L))$$

and

$$\varinjlim_I (G \circ F \circ L) \to G(F(\varinjlim_I (L)))$$

are isomorphisms. Hence, the canonical morphism

$$G(\varinjlim_I (F \circ L)) \to G(F(\varinjlim_I (L)))$$

is an isomorphism, too. But this morphism being the value under G of the canonical morphism $\varinjlim_I (F \circ L) \to F(\varinjlim_I (L))$ and G being faithful and exact, it follows that this last morphism is an isomorphism. □

(2.1.3) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Then, $\bullet_{[\psi]} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^H(R_{[\psi]})$ commutes with inductive limits and with finite projective limits.*

Proof. The forgetful functor $\mathsf{GrMod}^G(R) \to \mathsf{Mod}(R)$ is the composition of $\bullet_{[\psi]}$ with the forgetful functor $\mathsf{GrMod}^H(R_{[\psi]}) \to \mathsf{Mod}(R)$. As both these forgetful functors are faithful and commute with inductive limits and with finite projective limits by 2.1.1 b), the claim follows from 2.1.2 and its dual concerning projective limits. □

(2.1.4) Concerning set theory, we have to spell out the above more precisely. Proposition 2.1.1 is understood to say that $\mathsf{GrMod}^G(R)$ fulfils AB5 and AB4* with respect to $\mathcal{U}$ and that the forgetful functor $\mathsf{GrMod}^G(R) \to \mathsf{Mod}(R)$ and the functors $\bullet_g$ commute with $\mathcal{U}$-small limits. Hence, in the proof the set I has to be $\mathcal{U}$-small in all its occurences. Analogously, 2.1.3 is understood to say that $\bullet_{[\psi]}$ commutes with $\mathcal{U}$-small limits.

2.2. Free graded modules

Let G be a group, and let R be a G-graded ring.

In the category of modules over a ring, free modules are defined as values of a left adjoint of the forgetful functor to Ens. Here, we develop a graded analogue of this idea.

(2.2.1) There is a faithful functor

$$\mathsf{GrMod}^G(R) \to \mathsf{QGrEns}^{G,\mathrm{Id}_{\mathsf{Ens}}}$$

that maps a G-graded R-modules $(M, (M_g)_{g\in G})$ onto the G-quasigraded set $(\coprod_{g\in G} M_g, (M_g)_{g\in G})$. By abuse of language, we will call this *the forgetful functor from $\mathsf{GrMod}^G(R)$ to QGrEns^G.*

The above functor has a left adjoint

$$L : \mathsf{QGrEns}^{G,\mathrm{Id}_{\mathsf{Ens}}} \to \mathsf{GrMod}^G(R).$$

Indeed, we can construct L as follows. For a G-quasigraded set E, let $L(E)$ be the G-graded R-module $\bigoplus_{g\in G}(R(g)^{\oplus E_g})$. For a morphism of $u : E \to F$ in $\mathsf{QGrEns}^{G,\mathrm{Id}_{\mathsf{Ens}}}$, let $L(u) : L(E) \to L(F)$ be the morphism in $\mathsf{GrMod}^G(R)$ such that

$$L(u)_g : \bigoplus_{e\in E_g} R(g) \to \bigoplus_{f\in F_g} R(g)$$

is induced by the map $u_g : E_g \to F_g$ for every $g \in G$.

(2.2.2) If E is a $(G, \mathrm{Id}_{\mathsf{Ens}})$-quasigraded set, then the G-graded R-module $L(E)$ defined in 2.2.1 is called *the free G-graded R-module with basis E.* Clearly, it depends only on the $(G, \mathrm{Id}_{\mathsf{Ens}})$-quasigraded set $\coprod_{g\in G} \mathrm{Card}(E_g)$.

Now, let M be a G-graded R-module. If E is a $(G, \mathrm{Id}_{\mathsf{Ens}})$-quasigraded set, then M is called *free with basis E* if it is isomorphic to $L(E)$. Moreover, M is called *free* if there is a $(G, \mathrm{Id}_{\mathsf{Ens}})$-quasigraded set E such that M is free with basis E.

(2.2.3) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, let E be a $(G, \mathrm{Id}_{\mathsf{Ens}})$-quasigraded set, and let M be a G-graded R-module. If M is free with basis E, then $M_{[\psi]}$ is free with basis $E_{[\psi]}$. Indeed, this is clear by 2.2.1, 2.1.3 and 1.1.8.

(2.2.4) If, in the notations of 2.2.3, $M_{[\psi]}$ is free, then M is not necessarily free; for an example see [8, I.2.6.2].

(2.2.5) Let M be a free G-graded R-module. We define *the rank of M,* denoted by $\mathrm{rk}_R(M)$ or, if no confusion can arise, by $\mathrm{rk}(M)$, as the minimum of the cardinalities of all bases of M, where by the cardinality of a G-quasigraded set we mean the cardinality of its underlying set. If R is not the zero ring, then $\mathrm{rk}(M)$ equals the cardinality of every basis of M, as is seen from 2.2.3.

We end this section by the graded analogues of finitely generated and finitely presented modules.

(2.2.6) Let M be a G-graded R-module. Then, M is called *finitely generated* if there is an epimorphism $F \twoheadrightarrow M$ in $\mathsf{GrMod}^G(R)$ such that F is free of finite rank. This is the case if and only if the R-module underlying M has a finite homogeneous generating set, that is, a finite generating set contained in M^{hom}.

Now, let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Then, M is finitely generated if and only if $M_{[\psi]}$ is finitely generated. Indeed, if M is finitely generated, then 2.2.3 and 2.1.3 imply that $M_{[\psi]}$ is finitely generated, too. Conversely, if $M_{[\psi]}$ is finitely generated, then taking the homogeneous components with respect to the G-graduation of a finite homogeneous generating set of $M_{[\psi]}$ yields a finite homogeneous generating set of M.

(2.2.7) Let M be a G-graded R-module. A *presentation of* M is a pair $(u : F' \to F, v : F \to M)$ of morphisms in $\mathsf{GrMod}^G(R)$ such that F' and F are free and that the sequence $F' \xrightarrow{u} F \xrightarrow{v} M \to 0$ is exact. Clearly, there exists a presentation of M. The G-graded R-module M is called *finitely presented* if there is a presentation $(u : F' \to F, v : F \to M)$ of M such that F and F' are of finite rank.

Now, let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Then, M is finitely presented if and only if $M_{[\psi]}$ is finitely presented. Indeed, if M is finitely presented, then 2.2.3 and 2.1.3 imply that $M_{[\psi]}$ is finitely presented, too. Conversely, let $M_{[\psi]}$ be finitely presented. Then, the R-module underlying M is finitely presented by the above, and in particular finitely generated. Hence, there is a free G-graded R-module F of finite rank and an epimorphism $v : F \twoheadrightarrow M$ in $\mathsf{GrMod}^G(R)$, and it suffices to show that $\mathrm{Ker}(v)$ is finitely generated. By the above, it suffices to show that the R-module underlying $\mathrm{Ker}(v)$ is finitely generated. But this holds by [A, X.1.4 Proposition 6].

2.3. Hom functors

Let G be a group, and let R be a G-graded ring.

This section is devoted to graded analogues of Hom functors and tensor products, and especially their behaviour under coarsening functors. We start by defining graded Hom functors.

(2.3.1) Let M and N be G-graded R-modules. Let us temporarily denote by $H_g(M,N)$ the group $\mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M, N(g))$ for every $g \in G$, and by $H(M,N)$ the group $\bigoplus_{g \in G} H_g(M,N)$. For all $g,h \in G$, the structure of R-module on $\mathrm{Hom}_R(M,N)$ induces a biadditive map

$$R_g \times H_h(M,N) \to H_{g+h}(M,N).$$

These maps define a structure of G-graded R-module on $H(M,N)$, the G-graduation being given by $(H_g(M,N))_{g\in G}$. This G-graded R-module is denoted by ${}^G\mathrm{Hom}_R(M,N)$, and its component of degree $g \in G$ is denoted by ${}^g\mathrm{Hom}_R(M,N)$.

The above gives rise to a contra-covariant bifunctor

$$ {}^G\mathrm{Hom}_R(\bullet, \blacksquare) : \mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R). $$

(2.3.2) The contra-covariant bifunctor ${}^G\mathrm{Hom}_R(\bullet, \blacksquare)$ is left exact in both variables. Indeed, from its definition it is clear that it is additive in both arguments. Let $V : \mathsf{GrMod}^G(R) \to \mathsf{Mod}(R)$ denote the forgetful functor. As the contra-covariant bifunctor

$$ \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(\bullet, \blacksquare) : \mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R) \to \mathsf{Ab} $$

is left exact in both variables and as $\bullet(g)$ is an isomorphism of categories for every $g \in G$, the contra-covariant bifunctor

$$ {}^g\mathrm{Hom}_R(\bullet, \blacksquare) : \mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R) \to \mathsf{Ab} $$

is left exact in both variables. Since $\mathsf{Mod}(R)$ fulfils AB4, the contra-covariant biadditive functor

$$ V \circ {}^G\mathrm{Hom}_R(\bullet, \blacksquare) : \mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R) \to \mathsf{Mod}(R) $$

is left exact in both variables, too. Now, since V is faithful and exact by 2.1.1 b), and in particular additive, the claim follows from 2.1.2 and its dual.

(2.3.3) It holds ${}^G\mathrm{Hom}_R(R, \bullet) = \mathrm{Id}_{\mathsf{GrMod}^G(R)}$. Indeed, for a G-graded R-module M and $g \in G$ it holds $\mathrm{Hom}_{\mathsf{GrMod}^G(R)}(R, M(g)) = M_g$. Therefore, for a morphism u in $\mathsf{GrMod}^G(R)$ we have

$$ {}^G\mathrm{Hom}_R(R, u) = \bigoplus_{g\in G} \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(R, u(g)) = \bigoplus_{g\in G} u_g = u. $$

Now we will investigate the behaviour of graded Hom functors under coarsening. These functors do not commute in general, and so we look for conditions under which they do.

(2.3.4) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Let M and N be G-graded R-modules. By 1.1.8, for every $g \in G$ we get a monomorphism

$$ h_\psi(M,N)_g : {}^g\mathrm{Hom}_R(M,N) \rightarrowtail {}^{\psi(g)}\mathrm{Hom}_{R_{[\psi]}}(M_{[\psi]}, N_{[\psi]}),\ u \mapsto u_{[\psi]} $$

in Ab. These induce a monomorphism

$$ h_\psi(M,N) : {}^G\mathrm{Hom}_R(M,N)_{[\psi]} \rightarrowtail {}^H\mathrm{Hom}_{G_{[\psi]}}(M_{[\psi]}, N_{[\psi]}) $$

in $\mathsf{GrMod}^H(R_{[\psi]})$. Moreover, as this is natural in M and N, we have a monomorphism of contra-covariant bifunctors

$$ h_\psi : {}^G\mathrm{Hom}_R(\bullet, \blacksquare)_{[\psi]} \rightarrowtail {}^H\mathrm{Hom}_{R_{[\psi]}}(\bullet_{[\psi]}, \blacksquare_{[\psi]}). $$

(2.3.5) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in* Ab*, and let M be a G-graded R-module. If M is finitely generated, then*

$$h_\psi(M, \bullet) : {}^G\mathrm{Hom}_R(M, \bullet)_{[\psi]} \to {}^H\mathrm{Hom}_{R_{[\psi]}}(M_{[\psi]}, \bullet_{[\psi]})$$

is an isomorphism of functors.

Proof. Let M be finitely generated, and let N be a G-graded R-module. As $h_\psi(M, N)$ is a monomorphism by 2.3.4, it suffices to show that

$$h_\psi(M, N)_d : ({}^G\mathrm{Hom}_R(M, N)_{[\psi]})_d \to {}^d\mathrm{Hom}_{R_{[\psi]}}(M_{[\psi]}, N_{[\psi]})$$

is surjective for every $d \in H$. So, let $d \in H$ and let

$$u \in {}^d\mathrm{Hom}_{R_{[\psi]}}(M_{[\psi]}, N_{[\psi]}) = \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M_{[\psi]}, N_{[\psi]}(d)).$$

As M is finitely generated, there is a finite generating set $E \subseteq M^{\mathrm{hom}}$ of M.

Now, let $g \in G$. We claim that there is a morphism $v_g : M \to N$ in $\mathsf{Mod}(R)$ such that $v_g(e) = u(e)_{g+\deg(e)}$ for every $e \in E$. Indeed, let $(r_e)_{e\in E}$ be a family in R such that $\sum_{e\in E} r_e e = 0$. For every $f \in G$ it follows $0 = (\sum_{e\in E} r_e e)_{f-g} = \sum_{e\in E}(r_e)_{f-g-\deg(e)}e$ and therefore

$$0 = u(\textstyle\sum_{e\in E}(r_e)_{f-g-\deg(e)}e) = \sum_{e\in E}(r_e)_{f-g-\deg(e)}u(e).$$

Hence, we get

$$0 = (\textstyle\sum_{e\in E}(r_e)_{f-g-\deg(e)}u(e))_f = \sum_{e\in E}(r_e)_{f-g-\deg(e)}u(e)_{g+\deg(e)} =$$

$$\textstyle\sum_{e\in E}(r_e u(e)_{g+\deg(e)})_f = (\sum_{e\in E} r_e u(e)_{g+\deg(e)})_f$$

for every $f \in G$ and therefore $\sum_{e\in E} r_e u(e)_{g+\deg(e)} = 0$.

Moreover, for every $f \in G$ it holds $v_g(M_f) \subseteq N_{f+g}$, and hence we see that $v_g \in {}^g\mathrm{Hom}_R(M, N)$. If $v_g \neq 0$, then there is an $e \in E$ with $u(e)_{g+\deg(e)} \neq 0$. Hence, finiteness of E implies that the family $(v_g)_{g\in G}$ has finite support.

Finally, for every $e \in E$ it holds

$$\textstyle\sum_{g\in G} v_g(e) = \sum_{g\in G} u(e)_{g+\deg(e)} = u(e),$$

and this implies $u = \sum_{g\in G} v_g \in {}^G\mathrm{Hom}_R(M, N)$. □

(2.3.6) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. On use of [8, I.2.10] it can be shown that h_ψ is an isomorphism if $\mathrm{Ker}(\psi)$ is finite.

Having defined graded Hom functors, we can – as in the ungraded case – define graded tensor products as left adjoints thereof. We will show here that they exist, and moreover that they commute with coarsening.

(2.3.7) Let M be a G-graded R-module. Then, the functor

$${}^G\mathrm{Hom}_R(M, \bullet) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$$

has a left adjoint

$$T(\bullet, M) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R),$$

which we can construct as follows. For a G-graded R-module N, the R-module underlying $T(N,M)$ is $N \otimes_R M$, and its G-graduation is

$$\left(\bigoplus\{N_h \otimes_{R_0} M_{h'} \mid h, h' \in G \wedge h + h' = g\}\right)_{g\in G}.$$

For a morphism $u : N \to P$ in $\mathsf{GrMod}^G(R)$, the morphism $T(u,M) := u \otimes \mathrm{Id}_M$ in $\mathsf{Mod}(R)$ is a morphism $T(N,M) \to T(P,M)$ in $\mathsf{GrMod}^G(R)$. This gives rise to a functor

$$T(\bullet, M) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$$

with the desired property.

If we vary M we get a bifunctor

$$T(\bullet, \blacksquare) : \mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R),$$

called *the tensor product.* Moreover, $T(M, \bullet)$ and $T(\bullet, M)$ are canonically isomorphic for every G-graded R-module M, and hence $T(\bullet, \blacksquare)$ commutes with inductive limits in both arguments.

(2.3.8) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Then, it is clear from 2.3.7 that

$$T(\bullet, \blacksquare)_{[\psi]} = T(\bullet_{[\psi]}, \blacksquare_{[\psi]}).$$

In particular, the tensor product on $\mathsf{GrMod}^G(R)$ commutes with the forgetful functor from $\mathsf{GrMod}^G(R)$ to $\mathsf{Mod}(R)$. Therefore, no confusion will arise if we write $\bullet \otimes_R \blacksquare$ instead of $T(\bullet, \blacksquare)$.

(2.3.9) A G-graded R-module M is called *flat,* if the functor

$$M \otimes_R \bullet : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$$

is exact.

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab and M is a G-graded R-module such that $M_{[\psi]}$ is flat, then M is flat, too. Indeed, this follows from 2.1.3 and 2.3.8.

(2.3.10) In the notations of 2.3.9, one can show that if M is flat, then $M_{[\psi]}$ is flat, too (see [8, I.2.18]).

(2.3.11) Proposition *For every $g \in G$ there is a canonical isomorphism*

$$\bullet(g) \cong R(g) \otimes_R \bullet$$

of functors.

Proof. Let M be a G-graded R-module. The canonical isomorphism $M \cong R \otimes_R M$ in $\mathsf{Mod}(R)$ is an isomorphism in $\mathsf{GrMod}^G(R)$, and for every $h \in G$ it moreover induces an isomorphism

$$M(g)_h \cong (R \otimes_R M)_{h+g} = \bigoplus\{R_e \otimes_{R_0} M_f \mid e, f \in G \wedge e + f = h + g\} =$$

$$\bigoplus\{R(g)_e \otimes_{R_0} M_f \mid e, f \in G \wedge e + f = h\} = (R(g) \otimes_R M)_h$$

in $\mathsf{Mod}(R_0)$. These isomorphisms give rise to an isomorphism $M(g) \cong R(g) \otimes_R M$ that is clearly natural in M. $\square$

(2.3.12) Concerning set theory, we only have to consider 2.3.1. Since G and R are supposed to be elements of $\mathcal{U}$, the groups $H(M, N)$ are elements of $\mathcal{U}$ for all $M, N \in \mathrm{Ob}(\mathsf{GrMod}^G(R))$, and hence the bifunctor ${}^G\mathrm{Hom}_R(\bullet, \blacksquare)$ takes its values in $\mathsf{GrMod}^G(R)$.

2.4. Projective and injective graded modules

Let G be a group, and let R be a G-graded ring.

In this section we investigate projective and injective objects in categories of graded modules. We start by looking at their behaviour under coarsening, for which the following lemma will be helpful.

(2.4.1) Lemma *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, let M, N and L be G-graded R-modules, and let $u : M \to N$ be a morphism in $\mathsf{GrMod}^G(R)$. Moreover, let $v : M_{[\psi]} \to L_{[\psi]}$ and $w : L_{[\psi]} \to N_{[\psi]}$ be morphisms in $\mathsf{GrMod}^H(R_{[\psi]})$ such that $w \circ v = u_{[\psi]}$.*

a) If there exists a morphism $v' : M \to L$ in $\mathsf{GrMod}^G(R)$ with $v = v'_{[\psi]}$, then there exists a morphism $w' : L \to N$ in $\mathsf{GrMod}^G(R)$ with $w' \circ v' = u$.

b) If there exists a morphism $w' : L \to N$ in $\mathsf{GrMod}^G(R)$ with $w = w'_{[\psi]}$, then there exists a morphism of $v' : M \to L$ in $\mathsf{GrMod}^G(R)$ with $w' \circ v' = u$.

Proof. Let $v' \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M, L)$ be such that $v = v'_{[\psi]}$. Then, setting $w'_g : L_g \to N_g,\ x \mapsto w(x)_g$ for every $g \in G$, we get a morphism $w' : L \to N$ in $\mathsf{GrMod}^G(R)$. For $g \in G$ and $x \in L_g$ it holds $w'_{[\psi]}(v(x)) = w'_g(v(x)) = w(v(x))_g = u(x)_g = u_{[\psi]}(x)$. Hence, we get $w'_{[\psi]} \circ v = u_{[\psi]}$ and therefore $w' \circ v' = u$.

Conversely, let $w' \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(L, N)$ be such that $w = w'_{[\psi]}$. Then, setting $v'_g : M_g \to L_g,\ x \mapsto v(x)_g$ for every $g \in G$, we get a morphism $v' : M \to L$ in $\mathsf{GrMod}^G(R)$. For $g \in G$ and $x \in M_g$ it holds $w(v'_{[\psi]}(x)) = w(v'_g(x)) = w(v(x)_g) = w(v(x))_g = u(x)_g = u_{[\psi]}(x)$. Hence, we get $w \circ v'_{[\psi]} = u_{[\psi]}$ and therefore $w' \circ v' = u$. $\square$

(2.4.2) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, and let M be a G-graded R-module.*

a) If $M_{[\psi]}$ is projective, then so is M.

b) If $M_{[\psi]}$ is injective, then so is M.

Proof. This follows easily on use of 2.4.1. $\square$

(2.4.3) If, in the notations of 2.4.2, M is injective, then $M_{[\psi]}$ is not necessarily injective; for an example see [8, I.2.6.1]. For the converse of a) see 2.4.6.

(2.4.4) Corollary *Free G-graded R-modules are projective.*

Proof. The R-module underlying a free G-graded R-module is free by 2.2.3 and hence projective. Thus, 2.4.2 a) implies the claim. $\square$

Now, we will show that categories of graded modules have enough projective and enough injectives. The statement about injectives follows from general nonsense in Grothendieck's Tohoku paper [6], while the one about projectives is done directly after some easy preparations.

(2.4.5) Lemma *Let M be a G-graded R-module. Then, M is (finitely generated and) projective if and only if there is a free G-graded R-module L (of finite rank) and a left-invertible morphism $M \to L$ in $\mathsf{GrMod}^G(R)$.*

Proof. Using 2.2.1 and 2.4.4, this is straightforward. $\square$

(2.4.6) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, and let M be a G-graded R-module. Then, M is projective if and only if $M_{[\psi]}$ is projective.*

Proof. Let M be projective. Then, by 2.4.5 there is a free G-graded R-module L and a morphism $u : M \to L$ in $\mathsf{GrMod}^G(R)$ that has a left inverse v. Hence, we have a morphism $u_{[\psi]} : M_{[\psi]} \to L_{[\psi]}$ in $\mathsf{GrMod}^H(R_{[\psi]})$ and a left inverse $v_{[\psi]}$ of $u_{[\psi]}$. Thus, 2.2.3 and 2.4.5 imply that $M_{[\psi]}$ is projective. The converse holds by 2.4.2 a). $\square$

(2.4.7) Proposition *The G-graded R-module $\bigoplus_{g\in G} R(g)$ is a projective generator of $\mathsf{GrMod}^G(R)$.*

Proof. If M is a G-graded R-module, then the identity of the G-quasigraded set underlying M induces an epimorphism

$$\bigoplus_{g\in G}\bigoplus_{m\in M_g} R(g) \twoheadrightarrow M$$

in $\mathsf{GrMod}^G(R)$. Extending this by zero morphisms we get an epimorphism

$$\big(\bigoplus_{g\in G} R(g)\big)^{\oplus M^{\mathrm{hom}}} \twoheadrightarrow M$$

in $\mathsf{GrMod}^G(R)$. Hence, 2.1.1 a), [6, 1.9.1] and 2.4.4 imply the claim. $\square$

(2.4.8) Corollary *The category $\mathsf{GrMod}^G(R)$ has enough projectives and enough injectives.*

Proof. Having a projective generator by 2.4.7, it has enough projectives. Since it moreover fulfils AB5 by 2.1.1 a), it has enough injectives by [6, 1.10.1]. $\square$

(2.4.9) Proposition *Let M be a G-graded R-module.*

a) M is injective if and only if ${}^G\mathrm{Hom}_R(\bullet, M)$ is exact.

b) M is projective if and only if ${}^G\mathrm{Hom}_R(M, \bullet)$ is exact.

Proof. As $\mathsf{GrMod}^G(R)$ fulfils AB4 by 2.1.1 a), ${}^G\mathrm{Hom}_R(\bullet, M)$ is exact if and only if

$${}^g\mathrm{Hom}_R(\bullet, M) = \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(\bullet, M(g))$$

is exact for every $g \in G$. This is the case if and only if $M(g)$ is injective for every $g \in G$. As $\bullet(g)$ is an isomorphism of categories for every $g \in G$, this is equivalent to M being injective.

Analogously, ${}^G\mathrm{Hom}_R(M, \bullet)$ is exact if and only if

$${}^g\mathrm{Hom}_R(M, \bullet) = \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(\bullet(g), M)$$

is exact for every $g \in G$. As $\bullet(g)$ is an isomorphism of categories for every $g \in G$, this is the case if and only if $\mathrm{Hom}_{\mathsf{GrMod}^G(R)}(\bullet, M)$ is exact, that is, if and only if M is projective. □

(2.4.10) Proposition *A G-graded R-module M is injective if and only if for every $g \in G$, every monomorphism $v : N \rightarrowtail R(g)$ and every morphism $w : N \to M$ in $\mathsf{GrMod}^G(R)$ there is a morphism $u : R(g) \to M$ in $\mathsf{GrMod}^G(R)$ such that $u \circ v = w$.*

Proof. We set $P := \bigoplus_{g \in G} R(g)$. As $\mathsf{GrMod}^G(R)$ fulfils AB5 by 2.1.1 a) and as P is a generator of $\mathsf{GrMod}^G(R)$ by 2.4.7, we know from [6, 1.10 Lemme 1] that M is injective if and only if for every monomorphism $v : N \to P$ and every morphism $w : N \to M$ in $\mathsf{GrMod}^G(R)$ there exists a morphism $u : P \to M$ in $\mathsf{GrMod}^G(R)$ such that $u \circ v = w$.

Therefore, if M is injective and $g \in G$, then for every monomorphism $v : N \to R(g)$ and every morphism $w : N \to M$ in $\mathsf{GrMod}^G(R)$ there exists a morphism $u : P \to M$ in $\mathsf{GrMod}^G(R)$ such that $u \circ \iota_g \circ v = w$, where $\iota_g : R(g) \to P$ denotes the canonical injection.

Conversely, suppose that M fulfils the property stated in the claim. Let $v : N \rightarrowtail P$ be a monomorphism and let $w : N \to M$ be a morphism in $\mathsf{GrMod}^G(R)$. Then, for every $g \in G$ there exists a morphism $u_g : R(g) \to M$ in $\mathsf{GrMod}^G(R)$ such that $u_g \circ v_g = w \circ v'_g$, where v_g and v'_g respectively denote the canonical injections from $N \cap R(g)$ into N and $R(g)$. Hence, the family $(u_g)_{g \in G}$ induces a morphism $w : P \to M$ in $\mathsf{GrMod}^G(R)$ with $u \circ v = w$, and therefore M is injective. □

(2.4.11) Corollary *A G-graded R-module M is injective if and only if for every G-graded ideal $\mathfrak{a} \subseteq R$, every $g \in G$ and every morphism $v : \mathfrak{a} \to M(g)$ in $\mathsf{GrMod}^G(R)$ there is an $e \in M_g$ such that $h(a) = ae$ for every $a \in \mathfrak{a}$.*

Proof. This follows easily from 2.4.10. □

(2.4.12) Concerning set theory, the only point to consider is 2.4.7. Since we suppose that G and R are elements of $\mathcal{U}$ it is clear that $\bigoplus_{g \in G} R(g)$ is an element of $\mathcal{U}$, too.

2.5. Graded rings and modules of fractions

Let G be a group.

Formation of rings and modules of fractions can be carried out in the graded setting if the set of denominators consists of homogeneous elements. What is often more interesting than the whole graded rings and modules of fractions are their components of degree 0. We study these constructions in what follows.

(2.5.1) Let R be a G-graded ring, and let $S \subseteq R^{\text{hom}}$ be a subset. The multiplicative closure of S, denoted by $\overline{S}$, is again a subset of R^{hom}. Note that $\overline{\emptyset} = \{1\}$, the empty product being defined as the unit of R. We denote by $\widetilde{S}$ the homogeneous saturation of S, that is,

$$\widetilde{S} = \{r \in R^{\text{hom}} \mid \exists r' \in R : rr' \in \overline{S}\}.$$

Obviously, it holds $S \subseteq \overline{S} \subseteq \widetilde{S}$, as well as $\overline{\overline{S}} = \overline{S}$ and $\widetilde{\widetilde{S}} = \widetilde{\overline{S}} = \overline{\widetilde{S}} = \widetilde{S}$.

Now, let $T \subseteq R^{\text{hom}}$ be a further subset. If $S \subseteq T$, then it holds $\overline{S} \subseteq \overline{T}$ and $\widetilde{S} \subseteq \widetilde{T}$, and it holds $\overline{S} \subseteq \overline{T}$ if and only if $S \subseteq \overline{T}$. Moreover, the relations $\widetilde{S} \subseteq \widetilde{T}$, $\overline{S} \subseteq \widetilde{T}$ and $S \subseteq \widetilde{T}$ are equivalent.

If $S \subseteq \overline{T}$, then it holds $T \subseteq \widetilde{S}$ if and only if $\widetilde{S} = \widetilde{T}$. We say that T *is saturated over* S if $S \subseteq \overline{T} \subseteq \widetilde{S}$. This is the case if and only if $S \subseteq \overline{T}$ and $\widetilde{S} = \widetilde{T}$.

(2.5.2) We consider a category C^G defined as follows. The objects of C^G are pairs (R, S) consisting of a G-graded ring R and a subset $S \subseteq R^{\text{hom}}$; if (R, S) and (R', S') are objects of C^G, then a morphism in C^G from (R, S) to (R', S') is a morphism $u : R \to R'$ in GrAnn^G such that its underlying map induces by restriction and coastriction a map $\overline{S} \to \overline{S'}$; composition in C^G is induced by the composition of GrAnn^G.

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then there is a faithful functor $\bullet_{[\psi]} : \mathsf{C}^G \to \mathsf{C}^H$, mapping an object (R, S) onto $(R_{[\psi]}, S)$ and a morphism onto itself.

(2.5.3) There is a faithful functor $\mathsf{GrAnn}^G \to \mathsf{C}^G$, mapping a G-graded ring R onto $(R, R^* \cap R^{\text{hom}})$ and mapping a morphism in GrAnn^G onto itself. This functor has a left adjoint $F^G : \mathsf{C}^G \to \mathsf{GrAnn}^G$. Indeed, for $(R, S) \in \text{Ob}(\mathsf{C}^G)$ we define a G-graded ring $F^G(R, S)$, its underlying ring being the ring $S^{-1}R$ of fractions of R with denominators in S, and its G-graduation being given by

$$(S^{-1}R)_g = \{\tfrac{x}{s} \mid s \in \overline{S} \wedge x \in R^{\text{hom}} \wedge \deg(x) = \deg(s) + g\}$$

for every $g \in G$. For a morphism $u : (R, S) \to (R', S')$ in C^G, we define $F^G(u)$ as the morphism $S^{-1}R \to S'^{-1}R'$ in GrAnn^G with $\frac{x}{s} \mapsto \frac{u(x)}{u(s)}$ for $x \in R^{\text{hom}}$ and $s \in S$.

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then it holds

$$F^G(\bullet)_{[\psi]} = F^H(\bullet_{[\psi]}).$$

Therefore, for $(R,S) \in \mathrm{Ob}(\mathsf{C}^G)$, no confusion will arise if we denote $F^G(R,S)$ just by $S^{-1}R$.

Note that for $(R,S) \in \mathrm{Ob}(\mathsf{C}^G)$ it holds $S^{-1}R = \overline{S}^{-1}R$, and in particular for a G-graded ring R we have $\emptyset^{-1}R = R$.

(2.5.4) Let R be a G-graded ring, and let $S, T \subseteq R^{\mathrm{hom}}$. Then, Id_R is a morphism in C^G from (R,S) to (R,T) if and only if $S \subseteq \overline{T}$. In this case, by 2.5.3 we have a morphism $\eta^S_T(R) : S^{-1}R \to T^{-1}R$ in GrAnn^G with $\frac{x}{s} \mapsto \frac{x}{s}$ for $x \in R^{\mathrm{hom}}$ and $s \in S$. By 2.3.9 and [AC, II.2.3 Proposition 7; II.2.4 Théorème 1] it follows that $T^{-1}R$ is flat over $S^{-1}R$ by means of $\eta^S_T(R)$. If in addition T is saturated over S, then $\eta^S_T(R)$ is an isomorphism in GrAnn^G.

Now, let $U \subseteq R^{\mathrm{hom}}$ be a further subset, and suppose that $S \subseteq \overline{T}$, that $U \subseteq \overline{T}$, and that T is saturated over U. Then, by the above there is a morphism

$$(\eta^U_T(R))^{-1} \circ \eta^S_T(R) : S^{-1}R \to U^{-1}R$$

in GrAnn^G, denoted by abuse of language by $\eta^S_U(R)$.

Finally, for a subset $S \subseteq R^{\mathrm{hom}}$, we denote by $\eta_S(R)$ the morphism $\eta^{\emptyset}_S(R) : R \to S^{-1}R$ in GrAnn^G induced by $\mathrm{Id}_{(R,S)}$.

(2.5.5) Let R be a G-graded ring, and let $S \subseteq R^{\mathrm{hom}}$ be a subset. Then, by 2.5.4 we have the exact functor

$$\bullet \otimes_R S^{-1}R : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(S^{-1}R).$$

This commutes with coarsenings by 2.3.8 and 2.5.4. In particular, for a G-graded R-module M, the R-module underlying the G-graded R-module $S^{-1}R \otimes_R M$ is the R-module of fractions of M with denominators in S, and hence no confusion will arise if we denote $\bullet \otimes_R S^{-1}R$ just by $S^{-1}\bullet$. For a G-graded R-module M and $g \in G$ it holds

$$(S^{-1}M)_g = \{\tfrac{x}{s} \mid s \in \overline{S} \wedge x \in M^{\mathrm{hom}} \wedge \deg(x) = \deg(s) + g\}.$$

It follows from [A, II.5.1 Proposition 3] that there is a canonical isomorphism of bifunctors

$$(S^{-1}\bullet) \otimes_{S^{-1}R} (S^{-1}\,\blacksquare) \xrightarrow{\cong} S^{-1}(\bullet \otimes_R \blacksquare).$$

Furthermore, if $T, U \subseteq R^{\mathrm{hom}}$ are subsets with $S, U \subseteq \overline{T}$ such that T is saturated over U, we have the morphism of functors

$$\eta^S_U := \bullet \otimes_R \eta^S_U(R) : S^{-1}\bullet \to U^{-1}\bullet\,.$$

Moreover, we set $\eta_S := \eta^{\emptyset}_S : \mathrm{Id}_{\mathsf{GrMod}^G(R)} \to S^{-1}\bullet$.

Finally, for every $g \in G$ it holds $(S^{-1}\bullet)(g) = S^{-1}(\bullet(g))$.

(2.5.6) Let R be a G-graded ring, and let $S \subseteq R^{\mathrm{hom}}$ be a subset. We denote the ring $(S^{-1}R)_0$ by $(S)^{-1}R$ and the functor

$$(S^{-1}\bullet)_0 : \mathsf{GrMod}^G(R) \to \mathsf{Mod}((S)^{-1}R)$$

by $(S)^{-1}\bullet$. For every $g \in G$, it holds $(S)^{-1}(\bullet(g)) = (S^{-1}\bullet)_g$ by 2.5.5 and 1.1.8.

If $T, U \subseteq R^{\mathrm{hom}}$ are further subsets with $S, U \subseteq \overline{T}$ such that T is saturated over U, then we denote the morphisms $\eta_U^S(R)_0$ and $\eta_U^S(\bullet)_0$ respectively by $\eta_{(U)}^{(S)}(R)$ and $\eta_{(U)}^{(S)}$. Moreover, we write $\eta_{(S)}(R)$ and $\eta_{(S)}$ respectively for $\eta_{(S)}^{(\emptyset)}(R) = \eta_S(R)_0$ and $\eta_{(S)}^{(\emptyset)} = (\eta_S)_0$.

(2.5.7) Example Let R be a G-graded ring, and let $f \in R^{\mathrm{hom}}$. We consider $S = \{f\}$ and we write R_f, $R_{(f)}$, $\bullet_f$ and $\bullet_{(f)}$ respectively for $S^{-1}R$, $(S)^{-1}R$, $S^{-1}\bullet$ and $(S)^{-1}\bullet$. Moreover, let $g \in R^{\mathrm{hom}}$, and consider $U = \{g\}$ and $T = \{g, f\}$. Then, it holds $S, U \subseteq \overline{T}$, and T is saturated over U if and only if there exists $h \in R$ with $g = hf$. In this case we write η_g^f and $\eta_{(g)}^{(f)}$ respectively for η_U^S and $\eta_{(T)}^{(S)}$.

Now, let $g := \deg(f) \in G$ and $m, n \in \mathbb{Z}$. Then, the $R_{(f)}$-modules $(R(mg)_f)_{ng}$ and $R_{(f)}$ are isomorphic. Indeed, by 2.5.6 we can assume that $m = 0$, and then multiplication by $\frac{f^n}{1}$ induces an isomorphism as desired.

The next question addressed is if formation of components of degree 0 of modules of fractions commutes with tensor products. This is true for example if $G = \mathbb{Z}$ and moreover R is generated by its elements of degree 1 (see [ÉGA, II.2.5.13]), but not in general (see also IV.4.1.6 for an example). In particular, this observation is the reason for some differences in the behaviour of arbitrary toric schemes compared with the special case of projective spaces.

(2.5.8) Let R be a G-graded ring and let $S \subseteq R^{\mathrm{hom}}$. For G-graded R-modules M and N, the canonical injections $(S)^{-1}M \rightarrowtail S^{-1}M$ and $(S)^{-1}N \rightarrowtail S^{-1}N$ induce a morphism

$$(S)^{-1}M \otimes_{(S)^{-1}R} (S)^{-1}N \to S^{-1}M \otimes_{S^{-1}R} S^{-1}N$$

in $\mathsf{Mod}((S)^{-1}R)$ with $x \otimes y \mapsto x \otimes y$. Its composition with the canonical isomorphism

$$S^{-1}M \otimes_{S^{-1}R} S^{-1}N \xrightarrow{\cong} S^{-1}(M \otimes_R N)$$

in $\mathsf{GrMod}^G(R)$ from 2.5.5 is a morphism

$$\delta_S(M, N) : (S)^{-1}M \otimes_{(S)^{-1}(R)} (S)^{-1}N \to (S)^{-1}(M \otimes_R N)$$

in $\mathsf{Mod}((S)^{-1}R)$. This being natural in M and N, we get a morphism of bifunctors

$$\delta_S : ((S)^{-1}\bullet) \otimes_{(S)^{-1}R} ((S)^{-1}\blacksquare) \to (S)^{-1}(\bullet \otimes_R \blacksquare).$$

(2.5.9) Let R be a G-graded ring, and let $S, T, U \subseteq R^{\mathrm{hom}}$ be subsets such that $S, U \subseteq \overline{T}$ and that T is saturated over U. Then, the diagram of bifunctors

$$\begin{tikzcd}
((S)^{-1}\bullet) \otimes_{(S)^{-1}R} ((S)^{-1}\blacksquare) \arrow[r, "\delta_S"] \arrow[d, "\eta^{(S)}_{(U)} \otimes \eta^{(S)}_{(U)}"'] & (S)^{-1}(\bullet \otimes_R \blacksquare) \arrow[d, "\eta^{(S)}_{(U)}"] \\
((U)^{-1}\bullet) \otimes_{(U)^{-1}R} ((U)^{-1}\blacksquare) \arrow[r, "\delta_U"] & (U)^{-1}(\bullet \otimes_R \blacksquare)
\end{tikzcd}$$

commutes.

(2.5.10) Let R be a G-graded ring, let $S \subseteq R^{\mathrm{hom}}$ be a subset, and let I be a set. Then,

$$\delta_S(R^{\oplus I}, \bullet) : (S)^{-1}(R^{\oplus I}) \otimes_{(S)^{-1}R} ((S)^{-1}\bullet) \to (S)^{-1}(R^{\oplus I} \otimes_R \bullet)$$

is an isomorphism of functors. Indeed, $S^{-1}\bullet$ and $\bullet_0$ commute with inductive limits by 2.3.7 and 2.1.1 c) respectively, and from this the claim follows easily.

(2.5.11) Let $\varphi : F \rightarrowtail G$ be a monomorphism in Ab, let R be a G-graded ring, and let $S \subseteq R^{\mathrm{hom}}$ such that for every $s \in S$ it holds $\deg(s) \in \varphi(F)$. Then, S is a subset of $R_{(\varphi)}$, and it is easy to see that the F-graded $R_{(\varphi)}$-algebras $(\eta_S)_{(\varphi)} : R_{(\varphi)} \to (S^{-1}R)_{(\varphi)}$ and $\eta_S : R_{(\varphi)} \to S^{-1}(R_{(\varphi)})$ are canonically isomorphic; by means of this we will identify them.

(2.5.12) Concerning set theory, the only point to consider is 2.5.2. As GrAnn^G is a $\mathcal{U}$-category, the same holds for C^G, since the forgetful functor $\mathsf{C}^G \to \mathsf{GrAnn}^G$ is faithful.

3. Further properties of graded rings and modules

Let G be a group, and let R be a G-graded ring.

3.1. Strong graduations

In this section we define and characterise strongly graded rings. This will be used only in the study of flatness of sheaves on toric schemes at the end of Section IV.3.1. Extensive treatments of strong graduations can be found for example in [8] and [19].

(3.1.1) The G-graded ring R is called *strongly G-graded*, and its G-graduation is called *strong*, if $R_{g+h} = \langle R_g R_h \rangle_{\mathbb{Z}}$ for all $g, h \in G$, where $R_g R_h$ denotes the set of all elements of R of the form rs with $r \in R_g$ and $s \in R_h$.

(3.1.2) The functor $\bullet_0 : \mathsf{GrMod}^G(R) \to \mathsf{Mod}(R_0)$ equals the restriction functor with respect to the zero morphism $0 \rightarrowtail G$. By 1.4.8 it has a left adjoint $R \otimes_{R_0} \bullet$. Hence, there is a canonical morphism of functors

$$\nu : R \otimes_{R_0} (\bullet_0) \to \mathrm{Id}_{\mathsf{GrMod}^G(R)}.$$

For a G-graded R-module M, the morphism $\nu(M)$ in $\mathsf{GrMod}^G(R)$ is given by $r \otimes x \mapsto rx$.

(3.1.3) Proposition *The following conditions are equivalent:*

(i) R is strongly G-graded;

(ii) For every $g \in G$ it holds $1_R \in \langle R_g R_{-g} \rangle_{\mathbb{Z}}$;

(iii) The morphism of functors $\nu : R \otimes_{R_0} (\bullet_0) \to \mathrm{Id}_{\mathsf{GrMod}^G(R)}$ is an isomorphism.

Proof. If R is strongly G-graded, then condition (ii) is obviously fulfilled. So, assume that condition (ii) holds, and let M be a G-graded R-module. For all $g, h \in G$ we get

$$M_{g+h} = \langle 1_R M_{g+h} \rangle_{\mathbb{Z}} \subseteq \langle R_g R_{-g} M_{g+h} \rangle_{\mathbb{Z}} = \langle R_g M_h \rangle_{\mathbb{Z}} \subseteq M_{g+h}$$

and therefore $M_{g+h} = \langle R_g M_h \rangle_{\mathbb{Z}}$. Now, let $g \in G$ and $m \in M_g$. By the above, there are finite families $(r_i)_{i \in I}$ in R_g and $(m_i)_{i \in I}$ in M_0 such that $m = \sum_{i \in I} r_i m_i = \nu(M)(\sum_{i \in I} r_i \otimes m_i)$. Hence, $\nu(M)$ is an epimorphism. Clearly, it holds $\mathrm{Ker}(\nu(M))_0 = 0$, and for every $g \in G$ it holds $\mathrm{Ker}(\nu(M))_g = \langle R_g \,\mathrm{Ker}(\nu(M))_0 \rangle_{\mathbb{Z}} = 0$ by application of the above to $\mathrm{Ker}(\nu(M))$ instead of M. So, $\nu(M)$ is a monomorphism and therefore an isomorphism.

Finally, assume that ν is an isomorphism, and let $g, h \in G$. Then, $\nu(R(h)) : R \otimes_{R_0} (R(h)_0) \to R(h)$ is an isomorphism, and therefore

$$(\nu(R(h)))_g : R_g \otimes_{R_0} R_h \to R_{g+h}$$

is an isomorphism, too. This implies that R is strongly G-graded. □

3.2. Saturation

We introduce a graded version of saturation with respect to a graded ideal $\mathfrak{a}$, which coincides with the ungraded version. Moreover, we show that the formation of saturation with respect to $\mathfrak{a}$ is idempotent if $\mathfrak{a}$ is finitely generated.

(3.2.1) Let M be a G-graded R-module, and let $N \subseteq M$ be a G-graded sub-R-module. For a subset $U \subseteq R^{\mathrm{hom}}$, the set

$$(N :_M U) := \{x \in M \mid Ux \subseteq N\}$$

is a G-graded sub-R-module of M and equals $(N :_M \langle U\rangle_R)$.

Now, let $\mathfrak{a} \subseteq R$ be a G-graded ideal. By choosing a homogeneous generating set of $\mathfrak{a}$, the above implies that $(N :_M \mathfrak{a})$ is a G-graded sub-R-module of M. Moreover, it holds $N = (N :_M \mathfrak{a})$ if and only if $N = \bigcup_{n\in\mathbb{N}_0}(N :_M \mathfrak{a}^n)$, as is seen by induction on use of the relation

$$(N :_M \mathfrak{a}^n) = ((N :_M \mathfrak{a}^{n-1}) :_M \mathfrak{a}).$$

The set

$$\mathrm{Sat}_M(N, \mathfrak{a}) := \bigcup_{n\in\mathbb{N}_0} (N :_M \mathfrak{a}^n)$$

is a G-graded sub-R-module of M, called *the $\mathfrak{a}$-saturation of N in M*, and N is called *$\mathfrak{a}$-saturated in M* if $N = \mathrm{Sat}_M(N, \mathfrak{a})$.

If $\mathfrak{a}, \mathfrak{b} \subseteq R$ are G-graded ideals, by abuse of language the G-graded ideal $\mathfrak{b}^{\mathrm{sat},\mathfrak{a}} := \mathrm{Sat}_R(\mathfrak{b}, \mathfrak{a})$ of R is called *the $\mathfrak{a}$-saturation of* $\mathfrak{b}$, and $\mathfrak{b}$ is called *$\mathfrak{a}$-saturated* if it is $\mathfrak{a}$-saturated in R.

(3.2.2) Proposition *Let M be a G-graded R-module, let $N \subseteq M$ be a G-graded sub-R-module, and let $\mathfrak{a} \subseteq R$ be a finitely generated G-graded ideal. Then,* $\mathrm{Sat}_M(N, \mathfrak{a})$ *is the smallest G-graded sub-R-module of M that contains N and is $\mathfrak{a}$-saturated in M.*

Proof. By 3.2.1, $\mathrm{Sat}_M(N, \mathfrak{a})$ is a G-graded sub-R-module of M, and it holds

$$N = (N :_M \mathfrak{a}^0) \subseteq \mathrm{Sat}_M(N, \mathfrak{a}).$$

In particular, we get $\mathrm{Sat}_M(N, \mathfrak{a}) \subseteq \mathrm{Sat}_M(\mathrm{Sat}_M(N, \mathfrak{a}), \mathfrak{a})$. Now, let $x \in \mathrm{Sat}_M(\mathrm{Sat}_M(N, \mathfrak{a}), \mathfrak{a})$. There is an $n \in \mathbb{N}_0$ with $x \in (\mathrm{Sat}_M(N, \mathfrak{a}) :_M \mathfrak{a}^n)$, and this implies $\mathfrak{a}^n x \subseteq \mathrm{Sat}_M(N, \mathfrak{a})$. As $\mathfrak{a}$ is finitely generated, there is an $m \in \mathbb{N}_0$ with $\mathfrak{a}^{m+n}x \subseteq N$. From this it follows $x \in \mathrm{Sat}_M(N, \mathfrak{a})$. Thus, $\mathrm{Sat}_M(N, \mathfrak{a})$ is $\mathfrak{a}$-saturated in M.

Conversely, let $L \subseteq M$ be G-graded sub-R-module that is $\mathfrak{a}$-saturated in M and contains N, and let $x \in \mathrm{Sat}_M(N, \mathfrak{a})$. Then, there is an $n \in \mathbb{N}_0$ with $\mathfrak{a}^n x \subseteq N \subseteq L$, and hence it holds $x \in (L :_M \mathfrak{a}^n)$. As L is $\mathfrak{a}$-saturated in M, it follows $x \in L$, and from this we get the claim. □

3.3. Noetherianity

Also the notion of Noetherianity has a graded analogue that we look at next. This turns out to be a special case of the restricted Noetherianity properties introduced in I.1.3.10.

(3.3.1) A G-graded R-module M is called *Noetherian* if the set of subobjects of M, ordered by inclusion, is Noetherian. This is the case if and only if M it M^{hom}-Noetherian in the terminology from I.1.3.10. The G-graded ring R is called *Noetherian* if it is Noetherian considered as a G-graded R-module.

(3.3.2) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, and let M be a G-graded R-module. If $M_{[\psi]}$ is Noetherian, then M is Noetherian, too, since $\bullet_{[\psi]}$ is exact by 2.1.3.

(3.3.3) If, in the notations of 3.3.2, M is Noetherian, then $M_{[\psi]}$ is not necessarily Noetherian; for an example, consider $G = \mathbb{Z}^{\oplus \mathbb{N}_0}$ and the zero morphism $\psi : G \to 0$. Let K be a field, and let S be the Laurent algebra in countably many indeterminates $(X_i)_{i\in\mathbb{N}_0}$ over K, furnished with its canonical G-graduation. The only $\mathbb{Z}^{\oplus\mathbb{N}_0}$-graded ideals of S are 0 and S, and hence S is Noetherian. But $S_{[\psi]}$ is not Noetherian, as the ideal generated by $\{X_{2i} + X_{2i+1} \mid i \in \mathbb{N}_0\}$ is not finitely generated.

We ask now if Noetherianity is preserved under restriction functors. To find an answer we use the more general notion of restricted Noetherianity from I.1.3.10.

(3.3.4) Proposition *Let $\varphi : F \rightarrowtail G$ be a monomorphism in* Ab*, let M be a G-graded R-module, and let $\mathbb{L} \subseteq M^{\mathrm{hom}}$ be a subset.*

a) If the R-module M is $\mathbb{L}$-Noetherian, then the $R_{(\varphi)}$-module $M_{(\varphi)}$ is $\mathbb{L} \cap M_{(\varphi)}$-Noetherian.

b) If $N \subseteq M$ is a G-graded sub-R-module that is $\mathbb{L}$-generated as an R_0-module, then $N_{(\varphi)}$ is $\mathbb{L} \cap M_{(\varphi)}$-generated as an $R_{(\varphi)}$-module.

Proof. We set $\mathbb{L}_{(\varphi)} := \mathbb{L} \cap M_{(\varphi)}$. Suppose that M is $\mathbb{L}$-Noetherian, and let $(N_i)_{i\in\mathbb{N}_0}$ be an increasing sequence of $\mathbb{L}_{(\varphi)}$-generated and in particular F-graded sub-$R_{(\varphi)}$-modules of $M_{(\varphi)}$. Then, $(\langle N_i\rangle_R)_{i\in\mathbb{N}_0}$ is an increasing sequence of $\mathbb{L}$-generated sub-R-modules of M and therefore stationary. Hence, the sequence $(\langle N_i\rangle_R \cap M_{(\varphi)})_{i\in\mathbb{N}_0}$ is stationary, too, and by 1.4.11 it is equal to $(N_i)_{i\in\mathbb{N}_0}$. Thus, $M_{(\varphi)}$ is $\mathbb{L}_{(\varphi)}$-Noetherian.

Now, let $N \subseteq M$ be a G-graded sub-R-module that is $\mathbb{L}$-generated as an R_0-module, and let $f \in F$. Every element of $N_{\varphi(f)}$ is an R_0-linear combination of $\mathbb{L}_{(\varphi)}$, and hence $N_{(\varphi)}$ is $\mathbb{L}_{(\varphi)}$-generated as an R_0-module and therefore also as an $R_{(\varphi)}$-module. □

(3.3.5) Corollary *Let $\varphi : F \rightarrowtail G$ be a monomorphism in* Ab*, and let M be a Noetherian G-graded R-module. Then, the F-graded $R_{(\varphi)}$-module $M_{(\varphi)}$ is Noetherian.*

Proof. Clear from 3.3.1 and 3.3.4 a). □

(3.3.6) Corollary *Let M be a Noetherian G-graded R-module, and let $g \in G$. Then, the R_0-module M_g is Noetherian.*

Proof. If M is Noetherian, then so is $M(g)$. As $M_g \subseteq M^{\mathrm{hom}}$, we can apply 3.3.4 a) on $M(g)$ with $F = 0$ to get the claim. □

3.4. Hilbert's Basissatz and the Artin-Rees Lemma

The two results in the title are classical results about Noetherian rings and modules, and by means of the forgetful functor they can be applied to graded rings and modules. But if they are reproven in the right setting, that is, for categories of graded rings and modules, then we can replace the hypothesis of (ungraded) Noetherianity by graded Noetherianity. Such a hypothesis is on one hand more natural, because it concerns the "right" category, and on the other hand is strictly weaker than ungraded Noetherianity as we saw in 3.3.3.

We start by introducing graded versions of algebras of monoids. Special cases wil include graded polynomial rings.

(3.4.1) If $d : M \to G$ is a monoid over G, then we define a G-graded R-algebra $R[d]$, its underlying R-algebra being the algebra $R[M]$ of M over R and its G-graduation being given by

$$R[M]_g := \bigoplus_{h\in G} \bigoplus_{m\in d^{-1}(g-h)} (R_h \otimes_{R_0} R_0 e_m)$$

for every $g \in G$. If $e : N \to G$ is a further monoid over G and $u : M \to N$ is a morphism in $\mathsf{Mon}_{/G}$, then the morphism $R[u] : R[M] \to R[N]$ in $\mathsf{Alg}(R)$ is a morphism in $\mathsf{GrAlg}^G(R)$. This gives rise to a functor

$$R[\bullet] : \mathsf{Mon}_{/G} \to \mathsf{GrAlg}^G(R)$$

such that the diagram of categories

$$\begin{tikzcd} \mathsf{Mon}_{/G} \arrow[r, "R[\bullet]"] \arrow[d] & \mathsf{GrAlg}^G(R) \arrow[d] \\ \mathsf{Mon} \arrow[r, "R[\bullet]"] & \mathsf{Alg}(R), \end{tikzcd}$$

where the unmarked functors are the forgetful ones, commutes.

Now, let $d : M \to G$ be a monoid over G. The G-graded R-algebra $R[d]$ is called *the algebra of d over R* or, by abuse of language, *the G-graded algebra of M over R with respect to d.*

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then it holds

$$R[\bullet]_{[\psi]} = R_{[\psi]}[\psi \circ \bullet].$$

(3.4.2) Let $d : M \to G$ be a monoid over G. Clearly, the construction from 3.4.1 is functorial in R, that is, there is a functor

$$\bullet[d] : \mathsf{GrAnn}^G \to \mathsf{GrAnn}^G$$

under $\mathrm{Id}_{\mathsf{GrAnn}^G}$ such that the diagram of categories

$$\begin{tikzcd} \mathsf{GrAnn}^G \arrow[r, "{\bullet[d]}"] \arrow[d] & \mathsf{GrAnn}^G \arrow[d] \\ \mathsf{Ann} \arrow[r, "{\bullet[M]}"] & \mathsf{Ann}, \end{tikzcd}$$

where the unmarked functors are the forgetful ones, commutes.

Moreover, the above gives rise to a bifunctor

$$\bullet[\blacksquare] : \mathsf{GrAnn}^G \times \mathsf{Mon}_{/G} \to \mathsf{GrAnn}^G$$

under the canonical projection $\mathrm{pr}_1 : \mathsf{GrAnn}^G \times \mathsf{Mon}_{/G} \to \mathsf{GrAnn}^G$ such that the diagram of categories

$$\begin{tikzcd} \mathsf{GrAnn}^G \times \mathsf{Mon}_{/G} \arrow[r, "{\bullet[\blacksquare]}"] \arrow[d] & \mathsf{GrAnn}^G \arrow[d] \\ \mathsf{Ann} \times \mathsf{Mon} \arrow[r, "{\bullet[\blacksquare]}"] & \mathsf{Ann}, \end{tikzcd}$$

where the unmarked functors are the forgetful ones, commutes.

(3.4.3) Example Let I be a set, and let d be a map from I to the set underlying G. Then, d corresponds by [A, I.7.7 Proposition 10] to a unique monoid $d' : \mathbb{N}_0^{\oplus I} \to G$ over G. The algebra of d' over R is called *the G-graded polynomial algebra in the indeterminates $(X_i)_{i\in I}$ over R with respect to d* and is denoted by $R[(X_i)_{i\in I}, d]$. If $\mathrm{Card}(I) = 1$ and $d(I) = \{g\}$, then $R[(X_i)_{i\in I}, d]$ is, by abuse of language, denoted by $R[X, g]$ and called *the G-graded polynomial algebra in the indeterminate X over R with respect to g.*

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then it is clear from 3.4.1 that $R[(X_i)_{i\in I}, d]_{[\psi]} = R_{[\psi]}[(X_i)_{i\in I}, \psi \circ d]$. In particular, for $H = 0$ we see that the R-algebra underlying $R[(X_i)_{i\in I}, d]$ is the usual polynomial algebra in the indeterminates $(X_i)_{i\in I}$ over R, and for $i \in I$ it holds $\deg(X_i) = d(i)$. If moreover $\mathrm{Card}(I) = 1$ and $d(I) = \{g\}$, then it holds $R[X, g]_{[\psi]} = R_{[\psi]}[X, \psi(g)]$.

If S is a G-graded R-algebra, then there exist a set I, a map d from I to the set underlying G, and a surjective morphism $R[(X_i)_{i\in I}, d] \to S$ in $\mathsf{GrAlg}^G(R)$. If moreover S is finitely generated, then there exists such a map with a finite source.

(3.4.4) Let $\iota : 0 \rightarrowtail G$ denote the zero morphism in Ab, let $h : S \to T$ be a morphism in Ann, and let $d : M \to G$ be a monoid over G. Then, the diagram of categories

$$\begin{tikzcd}
\mathsf{Alg}(S) \arrow[r, "\bullet^{(\iota)}[d]"] \arrow[d, "\bullet\otimes_S T"'] & \mathsf{GrAlg}(S) \arrow[d, "\bullet\otimes_{S^{(\iota)}}(T)^{(\iota)}"] \\
\mathsf{Alg}(T) \arrow[r, "\bullet^{(\iota)}[d]"] & \mathsf{GrAlg}^G(T)
\end{tikzcd}$$

quasicommutes.

Having the Artin-Rees Lemma in view, we look now at graded variants of idealisation and of Rees algebras.

(3.4.5) Let M be a G-graded R-module. We define a G-graded R-algebra $I_G(M)$, its underlying R-algebra being the idealisation $R \oplus M$ of M and its G-graduation being given by $I_G(M)_g = R_g \oplus M_g$. By means of the canonical injection $M \rightarrowtail R \oplus M$, we may consider M as a G-graded ideal of $I_G(M)$, and we call $I_G(M)$ *the idealisation of M.*

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then it holds

$$I_G(M)_{[\psi]} = I_H(M_{[\psi]}).$$

So, no confusion will arise if we denote $I_G(M)$ again by $R \oplus M$.

(3.4.6) Let $\mathfrak{a} \subseteq R$ be a G-graded ideal, let $g \in G$, and consider the G-graded polynomial algebra $R[X, g]$ in one indeterminate X over R with respect to g. Then, the Rees algebra $R[\mathfrak{a}X]$ of $\mathfrak{a}$ over R is a G-graded sub-R-algebra of $R[X, g]$. As such it is denoted by $R[\mathfrak{a}, g]$ and called *the G-graded Rees algebra of $\mathfrak{a}$ over R with respect to g.*

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, it holds

$$R[\mathfrak{a}X, g]_{[\psi]} = R_{[\psi]}[\mathfrak{a}_{[\psi]}X, \psi(g)].$$

Now, we can prove the graded version of Hilbert's Basissatz. The proof is essentially the same as in the ungraded case. The crucial point is that besides the given graduation it makes use of another $\mathbb{Z}$-graduation – in the ungraded case these two coincide accidentally.

(3.4.7) Proposition *If the G-graded ring R is Noetherian, then every finitely generated G-graded R-algebra is Noetherian.*

Proof. If $f : S \to T$ is a surjective morphism in $\mathsf{GrAlg}^G(R)$ and if S is Noetherian, then it is easy to see that T is Noetherian, too. On use of this and of 3.4.3 it is clear that if suffices to show that for every $g \in G$ the G-graded polynomial ring $R[X, g]$ in one indeterminate X over R with respect to g is Noetherian. So, let $g \in G$, and set $S := R[X, g]$. Besides its G-graduation, we also furnish $R[X]$ with its canonical $\mathbb{Z}$-graduation and denote the corresponding degree map by $\deg_{\mathbb{Z}}$.

Now we assume that S is not Noetherian. Then, there is a G-graded ideal $\mathfrak{a} \subseteq S$ that is not finitely generated, and in particular it holds $\mathfrak{a} \neq 0$.

Hence, there exists $f_1 \in \mathfrak{a} \cap S^{\text{hom}} \setminus \{0\}$ such that $\deg_{\mathbb{Z}}(f_1)$ is minimal in $\{\deg_{\mathbb{Z}}(g) \mid g \in \mathfrak{a} \cap S^{\text{hom}} \setminus \{0\}\}$, and this implies $\langle f_1 \rangle_S \subsetneqq \mathfrak{a}$. By recursion we find a sequence $(f_i)_{i \in \mathbb{N}}$ in $\mathfrak{a} \cap S^{\text{hom}}$ such that for every $i \in \mathbb{N}$ we have $f_i \in \mathfrak{a} \cap S^{\text{hom}} \setminus \langle f_1, \ldots, f_{i-1} \rangle_S$ and that $\deg_{\mathbb{Z}}(f_i)$ is minimal in

$$\{\deg_{\mathbb{Z}}(g) \mid g \in \mathfrak{a} \cap S^{\text{hom}} \setminus \langle f_1, \ldots, f_{i-1} \rangle_S\}.$$

For every $i \in \mathbb{N}$, we set $d_i := \deg_{\mathbb{Z}}(f_i)$, and hence it holds $d_i \leq d_{i+1}$. For every $i \in \mathbb{N}$ there exist $a_i \in R^{\text{hom}} \setminus \{0\}$ and $g_i \in R[X]$ such that $\deg_{\mathbb{Z}}(g_i) < d_i$ and that $f_i = a_i X^{d_i} + g_i$. As R is Noetherian, the increasing sequence $(\langle a_1, \ldots, a_i \rangle_R)_{i \in \mathbb{N}}$ of G-graded ideals in R is stationary. Therefore, there is a $t \in \mathbb{N}$ with $a_{t+1} \in \langle a_1, \ldots, a_t \rangle_R$. Hence, there is a family $(c_i)_{i=1}^t$ in R^{hom} such that $a_{t+1} = \sum_{i=1}^t c_i a_i$. If we set $h := g_{t+1} - \sum_{i=1}^t c_i g_i X^{d_{t+1}-d_i}$, it follows

$$h = f_{t+1} - \sum_{i=1}^{t} c_i f_i X^{d_{t+1}-d_i} \in \mathfrak{a} \setminus \langle f_1, \ldots, f_t \rangle_S$$

and $\deg_{\mathbb{Z}}(h) < d_{t+1}$, which is a contradiction. □

(3.4.8) Corollary *Let the G-graded ring R be Noetherian.*

a) If M is a finitely generated G-graded R-module, then the idealisation of M is Noetherian.

b) If $\mathfrak{a} \subseteq R$ be a G-graded ideal and $g \in G$, then the G-graded Rees algebra of $\mathfrak{a}$ over R with respect to g is Noetherian.

Proof. Clear from 3.4.7. □

Finally we prove the graded Artin-Rees Lemma. The proof is the same as in the ungraded case, and we repeat it for the sake of completeness. It may be noted that in the statement no graduations are involved, but as said above the goal is to replace the (ungraded) Noetherianity hypothesis by the weaker and more natural graded Noetherianity hypothesis.

(3.4.9) Proposition *Let the G-graded ring R be Noetherian, let $\mathfrak{a} \subseteq R$ be a G-graded ideal, let M be a finitely generated G-graded R-module, and let $N \subseteq M$ be a G-graded sub-R-module. Then, there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds*

$$\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-n_0}(N \cap \mathfrak{a}^{n_0} M).$$

Proof. The idealisation of M is Noetherian by 3.4.8 a), and considering this we may assume that $\mathfrak{b} := N$ and $\mathfrak{c} := M$ are G-graded ideals of R with $\mathfrak{b} \subseteq \mathfrak{c}$. The G-graded Rees algebra of $\mathfrak{a}$ with respect to $0 \in G$ is Noetherian by 3.4.8 b). So, the intersection $\mathfrak{d}$ of the G-graded ideals $\mathfrak{b}R[X, g] \cap R[\mathfrak{a}X, g]$ and $\mathfrak{c}R[\mathfrak{a}X, g]$ of $R[\mathfrak{a}X, g]$ has a nonempty, finite, homogeneous generating set F. For every $f \in F$, there are a $d_f \in \mathbb{N}_0$ and a family $(a_i^{(f)})_{i=0}^{d_f}$ in R such that $f = \sum_{i=0}^{d_f} a_i^{(f)} X^i$, and this implies $a_i^{(f)} \in (\mathfrak{c}\mathfrak{a}^i) \cap \mathfrak{b}$ for every $f \in F$ and every $i \in [0, d_f]$. We set $n_0 := \max\{d_f \mid f \in F\}$, and we consider $n \geq n_0$ and $a \in (\mathfrak{c}\mathfrak{a}^n) \cap \mathfrak{b}$. Then, we have $aX^n \in \mathfrak{d}$, and hence there is a family $(g_f)_{f \in F}$ in

$R[\mathfrak{a}X, d]$ with $aX^n = \sum_{f\in F} g_f f$. For every $f \in F$, there is an $s_f \in \mathbb{N}_0$ and for every $j \in [0, s_f]$ there is a $b_j^{(f)} \in \mathfrak{a}^j$ such that $g_f = \sum_{j=0}^{s_f} b_j^{(f)} X^j$. From this we get that a is a sum of elements of the from $a_j^{(f)} b_k^{(f)}$ with $j \in [0, d_f]$ and $k \in [0, s_f]$ such that $j + k = n$. As such an element $a_j^{(f)} b_k^{(f)}$ is clearly contained in $\mathfrak{a}^{n-l}((\mathfrak{a}^j\mathfrak{c})\cap\mathfrak{b}) \subseteq \mathfrak{a}^{n-n_0}((\mathfrak{a}^{n_0}\mathfrak{c})\cap\mathfrak{b})$, we get $a \in \mathfrak{a}^{n-n_0}(\mathfrak{b}\cap(\mathfrak{a}^{n_0}\mathfrak{c}))$. Since the other inclusion is obvious, the claim is shown. □

(3.4.10) Concerning set theory, we have to consider 3.4.1 and 3.4.2. Since R and G are elements of $\mathcal{U}$ it follows from 1.2.6 that if M is an element of $\mathcal{U}$, then the same holds for $R[d]$. Moreover, for this condition to be fulfilled in the example 3.4.3 it suffices that the set I is $\mathcal{U}$-small.

3.5. Projective systems of ideals

In order to introduce torsion functors in the next section, we make some observations on projective systems of graded ideals of a graded ring. In particular, we show that whatever we do, we can assume such systems to be defined over ordered sets.

(3.5.1) We denote by $\mathsf{GrId}^G(R)$ the full subcategory of $\mathsf{GrMod}^G(R)_{/R}$ generated by all subobjects of R. By taking sources we can and do identify $\mathsf{GrId}^G(R)$ with a subcategory of $\mathsf{GrMod}^G(R)$.

(3.5.2) Let J be a category. Let J' be the quotient category of J defined by $\mathrm{Ob}(J') = \mathrm{Ob}(J)$ and by the coarsest equivalence relation on $\mathrm{Hom}_J(A, B)$ for all $A, B \in \mathrm{Ob}(J)$, that is, each two elements of $\mathrm{Hom}_J(A, B)$ are equivalent. Then, J' can be considered as a preordered set. Let J'' be the ordered set associated to J' (see [E, III.1.2]), considered as a category. The canonical morphism of preordered sets $J' \to J''$ is a functor, and composition with the canonical projection $J \to J'$ yields a functor $J \to J''$.

If $P : J \to \mathsf{GrId}^G(R)$ is a projective system in $\mathsf{GrId}^G(R)$ over J, then P factors uniquely over the above functor $J \to J''$, for $\mathsf{GrId}^G(R)$ is the category of subobjects of R in $\mathsf{GrMod}^G(R)$ and may therefore be considered as an ordered set. Moreover, if $u : P \to Q$ is a morphism of projective systems in $\mathsf{GrId}^G(R)$ over J, then it induces a unique morphism between the corresponding projective systems in $\mathsf{GrId}^G(R)$ over J''. From this we see that the categories of projective systems in $\mathsf{GrId}^G(R)$ over J and J'' respectively are isomorphic. Thus, studying projective systems in $\mathsf{GrId}^G(R)$ over arbitrary categories can be reduced to studying projective systems in $\mathsf{GrId}^G(R)$ over ordered sets.

(3.5.3) Let J be an ordered set. A projective system in $\mathsf{GrId}^G(R)$ over J is called a *projective system of G-graded ideals of R over J*, and if no confusion can arise we usually denote such a projective system in a form like $(\mathfrak{a}_j)_{j\in J}$. Then, for $j, k \in J$ with $j \leq k$ it holds $\mathfrak{a}_k \subseteq \mathfrak{a}_j$.

A projective system $(\mathfrak{a}_j)_{j\in J}$ of G-graded ideals of R over J is called *multiplicative,* if for all $j,k \in J$ there exists $l \in J$ with $j,k \leq l$ such that $\mathfrak{a}_l \subseteq \mathfrak{a}_j\mathfrak{a}_k$. In this case, the ordered set J is necessarily right filtering[1].

If $\mathfrak{A} = (\mathfrak{a}_j)_{j\in J}$ is a projective system of G-graded ideals of R over J, by taking cokernels we get a projective system $(R/\mathfrak{a}_j)_{j\in J}$ of G-graded R-modules over J, denoted by $R/\mathfrak{A}$.

(3.5.4) Example Let $\mathfrak{a} \subseteq R$ be a G-graded ideal. Then, $\mathfrak{A} := (\mathfrak{a}^n)_{n\in\mathbb{N}_0}$ is a multiplicative projective system of G-graded ideals of R over $\mathbb{N}_0$.

(3.5.5) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, let J be an ordered set, and let $\mathfrak{A} = (\mathfrak{a}_j)_{j\in J}$ be a projective system of G-graded ideals of R over J. The composition of $\mathfrak{A}$ with ψ-coarsening is a projective system of H-graded ideals of $R_{[\psi]}$ over J, denoted by $\mathfrak{A}_{[\psi]}$, and $\mathfrak{A}$ is multiplicative if and only if $\mathfrak{A}_{[\psi]}$ is multiplicative. Moreover, the composition of $R/\mathfrak{A}$ with ψ-coarsening is a projective system of H-graded $R_{[\psi]}$-modules over J, denoted by $(R/\mathfrak{A})_{[\psi]}$, and it holds $(R/\mathfrak{A})_{[\psi]} = R_{[\psi]}/\mathfrak{A}_{[\psi]}$.

3.6. Torsion functors

We define torsion functors on categories of graded modules with respect to projective systems of graded ideals, slightly generalising the definition in [2, 1.2.10–11].

(3.6.1) Let J be an ordered set, and let $\mathfrak{A} = (\mathfrak{a}_j)_{j\in J}$ be a projective system of G-graded ideals of R over J. For a G-graded R-module M, we set

$$^G\Gamma_{\mathfrak{A}}(M) := \bigcup_{j\in J}(0 :_M \mathfrak{a}_j).$$

This gives rise to a subfunctor $^G\Gamma_{\mathfrak{A}}$ of $\mathrm{Id}_{\mathsf{GrMod}^G(R)}$, called *the $\mathfrak{A}$-torsion functor.*

It holds $^G\Gamma_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} = {}^G\Gamma_{\mathfrak{A}}$, and a G-graded R-module M is called an *$\mathfrak{A}$-torsion module,* if $^G\Gamma_{\mathfrak{A}}(M) = M$. Clearly, if M is a G-graded R-module that is an $\mathfrak{A}$-torsion module, then every quotient and every subobject of M is an $\mathfrak{A}$-torsion module, too.

(3.6.2) Example Let $\mathfrak{a} \subseteq R$ be a G-graded ideal. If we consider the projective system $\mathfrak{A} = (\mathfrak{a}^n)_{n\in\mathbb{N}_0}$ of G-graded ideals of R over $\mathbb{N}_0$, we write $^G\Gamma_{\mathfrak{a}}$ instead of $^G\Gamma_{\mathfrak{A}}$ and call this *the $\mathfrak{a}$-torsion functor.* For a G-graded R-module M it holds $^G\Gamma_{\mathfrak{a}}(M) = \mathrm{Sat}_M(0,\mathfrak{a})$. Keeping in mind that the radical $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$ is a G-graded ideal of R, is is easily checked that if $\sqrt{\mathfrak{a}}$ is finitely generated, then $^G\Gamma_{\mathfrak{a}} = {}^G\Gamma_{\sqrt{\mathfrak{a}}}$.

In studying local cohomology (as we will do in the next section) an easy but fundamental result is that over Noetherian rings torsion functors

[1] An ordered set E is called *right filtering* if every nonempty finite subset of E has an upper bound in E.

conserve injectivity of modules (see [2, 2.1.4]). This property rather than Noetherianity makes some parts of the theory work as they do, and hence we will consider rings that have this so-called ITI-property.

(3.6.3) If J is an ordered set and $\mathfrak{A}$ is a projective system of G-graded ideals of R over J, we say that R has *the ITI-property with respect to* $\mathfrak{A}$ if ${}^G\Gamma_{\mathfrak{A}}(I)$ is injective for every injective G-graded R-module I. Moreover, if $\mathfrak{a} \subseteq R$ is a G-graded ideal, we say that R has *the ITI-property with respect to* $\mathfrak{a}$ if it has the ITI-property with respect to $(\mathfrak{a}^n)_{n\in\mathbb{N}_0}$. Finally, we say that R has *the ITI-property* if it has the ITI-property with respect to every G-graded ideal of R.

(3.6.4) Proposition *R has the ITI-property with respect to every finitely generated G-graded ideal of R if and only if it has the ITI-property with respect to every principal G-graded ideal of R.*

Proof. Suppose that R has the ITI-property with respect to every principal G-graded ideal of R, and let $\mathfrak{a} \subseteq R$ be a finitely generated G-graded ideal. Then there is a finite, homogeneous generating set E of $\mathfrak{a}$. We set $n := \mathrm{Card}(E)$ and prove by induction on n that R has the ITI-property with respect to $\mathfrak{a}$. For $n \in \{0,1\}$ this is clear. So, let $n > 1$ and assume the claim to be true for strictly smaller values of n. Let $a \in E$, and set $\mathfrak{b} := \langle E \setminus \{a\}\rangle_R$. Then it holds $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{b}+\langle a\rangle_R} = \Gamma_{\mathfrak{b}} \circ \Gamma_{\langle a\rangle_R}$, and this implies that R has the ITI-property with respect to $\mathfrak{a}$. □

(3.6.5) Proposition *Let J be an ordered set, and let $\mathfrak{A}$ be a multiplicative projective system of G-graded ideals of R over J. If R is Noetherian, then R has the ITI-property with respect to $\mathfrak{A}$.*

Proof. Let R be Noetherian, and let I be an injective G-graded R-module. We have to show that ${}^G\Gamma_{\mathfrak{A}}(I)$ is G-graded injective. Let $g \in G$, let $\mathfrak{b} \subseteq R$ be a G-graded ideal, and let $h : \mathfrak{b} \to {}^G\Gamma_{\mathfrak{A}}(I)(g)$ be a morphism in $\mathsf{GrMod}^G(R)$. By 2.4.11 it suffices to show that there is an $e \in {}^G\Gamma_{\mathfrak{A}}(I)_g$ such that $h(b) = be$ for every $b \in \mathfrak{b}$.

As I is injective, $I(g)$ is injective, too, and hence there is an $f \in I_g$ such that $h(b) = bf$ for every $b \in \mathfrak{b}$. In particular, $h(\mathfrak{b}) \subseteq \langle f\rangle_R$ is a G-graded sub-R-module, and it holds $h(\mathfrak{b}) \subseteq {}^G\Gamma_{\mathfrak{A}}(I)$. As $\mathfrak{b}$ is finitely generated and as $\mathfrak{A}$ is multiplicative, there is a $j \in J$ with $\mathfrak{a}_j h(\mathfrak{b}) = 0$.

As R is Noetherian and $\langle f\rangle_R$ is finitely generated, the Artin-Rees Lemma 3.4.9 implies the existence of an $n_0 \in \mathbb{N}$ such that

$$(\mathfrak{a}_j^n\langle f\rangle_R) \cap h(\mathfrak{b}) = \mathfrak{a}_j^{n-n_0}(\langle f\rangle_R \cap \mathfrak{a}_j^{n_0}h(\mathfrak{b}))$$

for every $n \geq n_0$. Applying this with $n = n_0 + 1$, we get

$$(\mathfrak{a}_j^{n_0+1}\langle f\rangle_R) \cap h(\mathfrak{b}) \subseteq \mathfrak{a}_j h(\mathfrak{b}) = 0.$$

As $\mathfrak{A}$ is multiplicative, there is a $k \in J$ with $j \leq k$ such that $\mathfrak{a}_k \subseteq \mathfrak{a}_j^{n_0+1}$, and hence we have $(\mathfrak{a}_k\langle f\rangle_R) \cap h(\mathfrak{b}) = 0$. In particular, there is the G-graded sub-R-module $\mathfrak{a}_k f \oplus \mathfrak{b}f \subseteq I$ and therefore the canonical projection $p : \mathfrak{a}_k f \oplus \mathfrak{b}f \to \mathfrak{b}f$. Hence, there is the morphism

$$h' : \mathfrak{a}_k + \mathfrak{b} \to (\mathfrak{b}f)(g),\ x \mapsto p(xf)$$

in $\mathsf{GrMod}^G(R)$.

As $I(g)$ is injective, there is an $e \in I_g$ such that it holds $h'(a) = ae$ for every $a \in \mathfrak{a}_k + \mathfrak{b}$. Hence, for every $a \in \mathfrak{a}_k$ it holds $ae = h'(a) = 0$. This yields $\mathfrak{a}_k e$ and therefore $e \in {}^G\Gamma_{\mathfrak{A}}(I)_g$. As $h(b) = bf = h'(b) = be$ for every $b \in \mathfrak{b}$, this implies the claim. □

(3.6.6) Corollary *If R is Noetherian, then it has the ITI-property.*

Proof. Clear from 3.5.4 and 3.6.5. □

4. Cohomology of graded modules

Let G be a group, and let R be a G-graded ring.

4.1. Complexes and cocomplexes

We introduce notations and terminology about complexes and cocomplexes (and by the way see that these are examples of quasigraded objects), and we look at their behaviour under coarsening functors.

(4.1.1) The category $\mathsf{Co}(\mathsf{GrMod}^G(R))$ of complexes in $\mathsf{GrMod}^G(R)$ has as objects $(\mathbb{Z}, \mathrm{Id}_{\mathsf{GrMod}^G(R)})$-quasigraded objects C in $\mathsf{GrMod}^G(R)$ furnished with an endomorphism $C \to C(-1)$ (where the shift is meant with respect to the $\mathbb{Z}$-quasigraduation), and the category $\mathsf{CCo}(\mathsf{GrMod}^G(R))$ of cocomplexes in $\mathsf{GrMod}^G(R)$ has as objects $(\mathbb{Z}, \mathrm{Id}_{\mathsf{GrMod}^G(R)})$-quasigraded objects C in $\mathsf{GrMod}^G(R)$ furnished with an endomorphism $C \to C(1)$. Clearly, coarsening with respect to the $\mathbb{Z}$-quasigraduations by the isomorphism of groups $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto -n$ induces an isomorphism between $\mathsf{Co}(\mathsf{GrMod}^G(R))$ and $\mathsf{CCo}(\mathsf{GrMod}^G(R))$.

It follows from 2.1.1 a) and [6, 1.6.1; 1.7 e)] that $\mathsf{Co}(\mathsf{GrMod}^G(R))$ and $\mathsf{CCo}(\mathsf{GrMod}^G(R))$ are Abelian categories fulfilling AB5 and AB4*.

(4.1.2) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Then, ψ-coarsening on $\mathsf{GrMod}^G(R)$ induces by [6, 1.6] functors

$$\mathsf{Co}(\mathsf{GrMod}^G(R)) \to \mathsf{Co}(\mathsf{GrMod}^H(R_{[\psi]}))$$

and

$$\mathsf{CCo}(\mathsf{GrMod}^G(R)) \to \mathsf{CCo}(\mathsf{GrMod}^H(R_{[\psi]})),$$

both of which are denoted again by $\bullet_{[\psi]}$ and called *the ψ-coarsening.* It is easy to see that they are faithful and commute with inductive limits and with finite projective limits.

(4.1.3) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, and let $i \in \mathbb{Z}$. It follows from 4.1.2 that the diagrams of categories

$$\begin{tikzcd}
\mathsf{Co}(\mathsf{GrMod}^G(R)) \arrow[r, "H_i"] \arrow[d, "\bullet_{[\psi]}"'] & \mathsf{GrMod}^G(R) \arrow[d, "\bullet_{[\psi]}"] \\
\mathsf{Co}(\mathsf{GrMod}^H(R_{[\psi]})) \arrow[r, "H_i"] & \mathsf{GrMod}^H(R_{[\psi]})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{CCo}(\mathsf{GrMod}^G(R)) \arrow[r, "H^i"] \arrow[d, "\bullet_{[\psi]}"'] & \mathsf{GrMod}^G(R) \arrow[d, "\bullet_{[\psi]}"] \\
\mathsf{CCo}(\mathsf{GrMod}^H(R_{[\psi]})) \arrow[r, "H^i"] & \mathsf{GrMod}^H(R_{[\psi]})
\end{tikzcd}$$

commute, that is, homology and cohomology commute with ψ-coarsening.

(4.1.4) Concerning set theory, we have to consider 4.1.1 and 4.1.2. If $\mathsf{GrMod}^G(R)$ is a $\mathcal{U}$-category, then so are $\mathsf{Co}(\mathsf{GrMod}^G(R))$ and $\mathsf{CCo}(\mathsf{GrMod}^G(R))$. Indeed, this follows from 1.1.9, the forgetful functors from $\mathsf{Co}(\mathsf{GrMod}^G(R))$ and $\mathsf{CCo}(\mathsf{GrMod}^G(R))$ respectively to $\mathsf{QGr}(\mathsf{GrMod}^G(R))^{\mathbb{Z}}$ being faithful. Furthermore, the last statement in 4.1.1 means that $\mathsf{Co}(\mathsf{GrMod}^G(R))$ and $\mathsf{CCo}(\mathsf{GrMod}^G(R))$ fulfil the axioms AB5 and AB4* with respect to $\mathcal{U}$, and the last statement in 4.1.2 means that coarsening commutes with $\mathcal{U}$-small inductive limits.

4.2. Resolutions

(4.2.1) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab and let M be a G-graded R-module. Let N be a complex in $\mathsf{GrMod}^G(R)$ and let $u : N \to M$ be a morphism in $\mathsf{Co}(\mathsf{GrMod}^G(R))$, where we consider M as a complex with differential 0. As $\bullet_{[\psi]}$ is exact and faithful by 4.1.2, u is a left resolution of M if and only if $u_{[\psi]}$ is a left resolution of $M_{[\psi]}$. Moreover, 2.4.6 implies that u is a projective left resolution of M if and only if $u_{[\psi]}$ is a projective left resolution of $M_{[\psi]}$.

Analogously, if N is a cocomplex in $\mathsf{GrMod}^G(R)$ and $u : M \to N$ is a morphism in $\mathsf{CCo}(\mathsf{GrMod}^G(R))$, then u is a right resolution of M if and only if $u_{[\psi]}$ is a right resolution of $M_{[\psi]}$.

Besides the ITI-property, local cohomology uses a further fundamental property, namely, that torsion modules over Noetherian rings have injective resolutions consisting of torsion modules (see [2, 2.1.6]). We call this the ITR-property.

(4.2.2) Let J be an ordered set, and let $\mathfrak{A}$ be a projective system of G-graded ideals of R over J. If M is a G-graded R-module, a (left or right) resolution N of M is called an $\mathfrak{A}$-*torsion resolution* if every component of N is an $\mathfrak{A}$-torsion module.

We say that R has *the ITR-property with respect to* $\mathfrak{A}$ if every G-graded R-module that is an $\mathfrak{A}$-torsion module has an injective $\mathfrak{A}$-torsion right resolution. Moreover, if $\mathfrak{a} \subseteq R$ is a G-graded ideal, we say that R has *the ITR-property with respect to* $\mathfrak{a}$ if it has the ITR-property with respect to $(\mathfrak{a}^n)_{n \in \mathbb{N}_0}$. Finally, we say that R has *the ITR-property* if it has the ITR-property with respect to every G-graded ideal of R.

(4.2.3) Proposition *Let J be an ordered set, and let $\mathfrak{A}$ be a projective system of G-graded ideals of R over J. If R has the ITI-property with respect to $\mathfrak{A}$, then it has the ITR-property with respect to $\mathfrak{A}$.*

PROOF. Straightforward by recursion on use of 2.4.8 and 3.6.1. □

(4.2.4) Corollary *If R is Noetherian, then it has the ITR-property.*

PROOF. Clear from 3.6.6 and 4.2.3. □

4.3. Extension functors

Now we are ready to start with cohomology. We use the knowledge gained in Section 2 about categories of graded modules to get graded Ext functors, defined as right derived cohomological functors of graded Hom functors. The main point in our approach is that we carry out everything within the category $\mathsf{GrMod}^G(R)$ instead of meddling with forgetful functors.

(4.3.1) From 2.4.8 we know that the category $\mathsf{GrMod}^G(R)$ has enough projectives and enough injectives. Moreover, the contra-covariant bifunctor ${}^G\mathrm{Hom}_R(\bullet, \blacksquare)$ is left exact in both arguments by 2.3.2 and in particular additive. Finally, ${}^G\mathrm{Hom}_R(\bullet, I)$ is exact if I is an injective G-graded R-module, and ${}^G\mathrm{Hom}_R(P, \bullet)$ is exact if P is a projective G-graded R-module by 2.4.9 b). Hence, it follows from [6, 2.3] that there is a sequence $({}^G\mathrm{Ext}^i_R(\bullet, \blacksquare))_{i\in\mathbb{Z}}$ of contra-covariant bifunctors from $\mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$ and appropriate connecting morphisms such that $({}^G\mathrm{Ext}^i_R(\bullet, M))_{i\in\mathbb{Z}}$ is the right derived cohomological functor of ${}^G\mathrm{Hom}_R(\bullet, M)$ and $({}^G\mathrm{Ext}^i_R(M, \bullet))_{i\in\mathbb{Z}}$ is the right derived cohomological functor of ${}^G\mathrm{Hom}_R(M, \bullet)$ for every G-graded R-module M.[2] Moreover, these δ-functors are universal, and it holds ${}^G\mathrm{Ext}^0_R(\bullet, \blacksquare) = {}^G\mathrm{Hom}_R(\bullet, \blacksquare)$.

Naturally, the first question about graded Ext functors is if they commute with coarsening – including as a special case the forgetful functor that forgets the graduation. The work on coarsening and graded Hom functors in 2.3 combined with δ-functor techniques gives us some answers.

(4.3.2) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, and let M be a G-graded R-module. As $\bullet_{[\psi]}$ is exact, it is clear that $({}^G\mathrm{Ext}^i_R(M, \bullet)_{[\psi]})_{i\in\mathbb{Z}}$ is an exact δ-functor and that ${}^G\mathrm{Ext}^i_R(M, \bullet)_{[\psi]}$ is effaceable for every $i \in \mathbb{N}$. Hence, [6, 2.2.1] implies that $({}^G\mathrm{Ext}^i_R(M, \bullet)_{[\psi]})_{i\in\mathbb{Z}}$ is a universal δ-functor, namely the right derived cohomological functor of ${}^G\mathrm{Hom}_R(M, \bullet)_{[\psi]}$.

On the other hand, $({}^H\mathrm{Ext}^i_{R_{[\psi]}}(M_{[\psi]}, \bullet_{[\psi]}))_{i\in\mathbb{Z}}$ is an exact δ-functor, too. Therefore, universality and 2.3.4 imply the existence of a unique morphism of δ-functors

$$(h^i_\psi(M, \bullet))_{i\in\mathbb{Z}} : \big({}^G\mathrm{Ext}^i_R(M, \bullet)_{[\psi]}\big)_{i\in\mathbb{Z}} \to \big({}^H\mathrm{Ext}^i_{R_{[\psi]}}(M_{[\psi]}, \bullet_{[\psi]})\big)_{i\in\mathbb{Z}}$$

with $h^0_\psi(M, \bullet) = h_\psi(M, \bullet)$.

(4.3.3) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, and let M be a G-graded R-module. As $\bullet_{[\psi]}$ is exact, it is clear that $({}^G\mathrm{Ext}^i_R(\bullet, M)_{[\psi]})_{i\in\mathbb{Z}}$ is an exact contravariant δ-functor and that ${}^G\mathrm{Ext}^i_R(\bullet, M)_{[\psi]}$ is effaceable for every $i \in \mathbb{N}$. Hence, [6, 2.2.1] implies that $({}^G\mathrm{Ext}^i_R(\bullet, M)_{[\psi]})_{i\in\mathbb{Z}}$ is a universal contravariant δ-functor, namely the right derived cohomological functor of ${}^G\mathrm{Hom}_R(\bullet, M)_{[\psi]}$.

[2] By abuse of language we denote a δ-functor as a sequence, its connecting morphisms being tacitly understood.

On the other hand, $({}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, M_{[\psi]}))_{i\in\mathbb{Z}}$ is an exact contravariant δ-functors, too, and using 2.4.6 we see that ${}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, M_{[\psi]})$ is effaceable for every $i \in \mathbb{N}$. Hence, it follows from [6, 2.2.1] that $({}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, M_{[\psi]}))_{i\in\mathbb{Z}}$ is a universal contravariant δ-functor, namely the right derived cohomological functor of ${}^H\mathrm{Hom}_{R_{[\psi]}}(\bullet_{[\psi]}, M_{[\psi]})$. Moreover, universality and 2.3.4 imply that there is a unique morphism of contravariant δ-functors

$$(h^i_\psi(\bullet, M))_{i\in\mathbb{Z}} : \big({}^G\mathrm{Ext}^i_R(\bullet, M)_{[\psi]}\big)_{i\in\mathbb{Z}} \to \big({}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, M_{[\psi]})\big)_{i\in\mathbb{Z}}$$

with $h^0_\psi(\bullet, M) = h_\psi(\bullet, M)$.

(4.3.4) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Combining 4.3.2 and 4.3.3, and keeping in mind [6, 2.3], we see that for every $i \in \mathbb{Z}$ there is a morphism of contra-covariant bifunctors

$$h^i_\psi : {}^G\mathrm{Ext}^i_R(\bullet, \blacksquare)_{[\psi]} \to {}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, \blacksquare_{[\psi]})$$

with $h^0_\psi = h_\psi$.

(4.3.5) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. Using 4.3.3, universality implies that if M is a G-graded R-module such that $h_\psi(\bullet, M)$ is an isomorphism of contravariant functors, then $(h^i_\psi(\bullet, M))_{i\in\mathbb{Z}}$ is an isomorphism of contravariant δ-functors. Hence, from 4.3.4 we see that if h_ψ is an isomorphism of contra-covariant bifunctors, then h^i_ψ is an isomorphism of contra-covariant bifunctors for every $i \in \mathbb{Z}$.

(4.3.6) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, and let M be a G-graded R-module. If M has a finitely generated projective resolution*[3]*, then $(h^i_\psi(M, \bullet))_{i\in\mathbb{Z}}$ is an isomorphism of δ-functors.*

Proof. Let P be a finitely generated projective resolution of M. Let N be a G-graded R-module, and let $i \in \mathbb{Z}$. As every component of P is finitely generated, it follows from 2.3.5 that $h_\psi(P, N)$ is an isomorphism in $\mathsf{Co}(\mathsf{GrMod}^H(R_{[\psi]}))$, and therefore $H^i(h_\psi(P, N))$ is an isomorphism in $\mathsf{GrMod}^H(R_{[\psi]})$. Thus, 4.3.1 and 4.3.4 show that $h^i_\psi(M, N) = H^i(h_\psi(P, N))$ is an isomorphism in $\mathsf{GrMod}^H(R_{[\psi]})$, and this implies the claim. □

(4.3.7) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab. According to 4.3.5 and 2.3.6, if $\mathrm{Ker}(\psi)$ is finite, then the morphism h^i_ψ is an isomorphism for every $i \in \mathbb{Z}$.

Now we combine graded Ext functors with inductive limits and hence prepare the ground for defining local cohomology and ideal transformations in the following section.

(4.3.8) Let J be a right filtering ordered set, let $F : J \to \mathsf{GrMod}^G(R)$ be a projective system in $\mathsf{GrMod}^G(R)$ over J, and let $i \in \mathbb{Z}$. Then, composition

[3] By abuse of language, a left resolution of a G-graded R-module is called *finitely generated* if every component of its source is finitely generated.

with the contravariant functor

$$\mathsf{GrMod}^G(R) \to \mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$$

which maps a G-graded R-module M onto the functor ${}^G\mathrm{Ext}^i_R(M, \bullet)$, yields a functor

$$J \to \mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$$

which maps an element $j \in J$ onto the functor ${}^G\mathrm{Ext}^i_R(F(j), \bullet)$. As the Abelian category $\mathsf{GrMod}^G(R)$ fulfils AB5 by 2.1.1 a), the same is true by [6, 1.6.1; 1.7 d)] for $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$. Hence, we can take the inductive limit in $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$ over J of the above functor to get a functor

$$\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R).$$

For $i = 0$, the functor $\varinjlim_J {}^G\mathrm{Ext}^0_R(F, \bullet) = \varinjlim_J {}^G\mathrm{Hom}_R(F, \bullet)$ is left exact, since $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$ fulfils AB5. It is readily checked that if we take the right derived cohomological functor of this functor, then the resulting universal δ-functor is $(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet))_{i\in\mathbb{Z}}$.

(4.3.9) Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, let J be a right filtering ordered set, and let $F : J \to \mathsf{GrMod}^G(R)$ be a projective system in $\mathsf{GrMod}^G(R)$ over J. Then, composition with ψ-coarsening yields a projective system $F(\bullet)_{[\psi]} : J \to \mathsf{GrMod}^H(R_{[\psi]})$ in $\mathsf{GrMod}^H(R_{[\psi]})$ over J. Moreover, the morphism h_ψ from 2.3.4 induces a monomorphism of functors

$$h_{F,\psi} := \varinjlim_J h_\psi(F, \bullet) : \varinjlim_J {}^G\mathrm{Hom}_R(F, \bullet)_{[\psi]} \rightarrowtail \varinjlim_J {}^H\mathrm{Hom}_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]}).$$

Exactness of ψ-coarsening gives rise to the universal δ-functor

$$\big(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)_{[\psi]}\big)_{i\in\mathbb{Z}}$$

and the exact δ-functor

$$\big(\varinjlim_J {}^H\mathrm{Ext}^i_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]})\big)_{i\in\mathbb{Z}},$$

and universality yields a unique morphism of δ-functors

$$(h^i_{F,\psi})_{i\in\mathbb{Z}} : \big(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)_{[\psi]}\big)_{i\in\mathbb{Z}} \to \big(\varinjlim_J {}^H\mathrm{Ext}^i_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]})\big)_{i\in\mathbb{Z}}$$

such that $h^0_{F,\psi} = h_{F,\psi}$.

(4.3.10) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in Ab, let J be a right filtering ordered set, and let $F : J \to \mathsf{GrMod}^G(R)$ be a projective system in $\mathsf{GrMod}^G(R)$ over J.*

a) If $F(j)$ is finitely generated for every $j \in J$, then the morphism of functors $h^0_{F,\psi}$ is an isomorphism.

b) If $F(j)$ has a finitely generated projective resolution for every $j \in J$, then the morphism of δ-functors $(h^i_{F,\psi})_{i\in\mathbb{Z}}$ is an isomorphism.

Proof. Claim a) is clear by 2.3.5. Concerning claim b), it suffices to show that $(\varinjlim_J {}^H\mathrm{Ext}^i_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]}))_{i\in\mathbb{Z}}$ is a universal δ-functor. Our hypothesis on F and 4.3.6 yield for every $i \in \mathbb{Z}$ an isomorphism

$$\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)_{[\psi]} \cong \varinjlim_J {}^H\mathrm{Ext}^i_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]}).$$

As the latter functor is effaceable for every $i \in \mathbb{N}$, the claim follows from [6, 2.2.1]. $\square$

(4.3.11) Concerning set theory, we have to consider 4.3.8, 4.3.9 and 4.3.10. There, we have to suppose J to be $\mathcal{U}$-small and $\mathsf{GrMod}^G(R)$ to be a $\mathcal{U}$-category (as holds if R and G are elements of $\mathcal{U}$ as supposed throughout). Then, $\mathsf{GrMod}^G(R)$ fulfils AB5 with respect to $\mathcal{U}$. Moreover, $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$ is not necessarily a $\mathcal{U}$-category, but it fulfils AB5 with respect to $\mathcal{U}$. As J is $\mathcal{U}$-small, this is what we needed in 4.3.8 to define the functors $\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)$.

4.4. Local cohomology and higher ideal transformation

Let J be a right filtering ordered set, and let $\mathfrak{A} = (\mathfrak{a}_j)_{j\in J}$ be a projective system of G-graded ideals of R over J.

We define local cohomology functors and higher ideal transformation functors as inductive limits of certain Ext functors. For local cohomology this is in contrast with the approach in [2], but we will see soon that it lead to the same, namely, the right derived cohomological functors of torsion functors.

(4.4.1) We apply the construction in 4.3.8 to the projective systems $\mathfrak{A}$ and $R/\mathfrak{A}$ in $\mathsf{GrMod}^G(R)$ over R, and we set

$${}^GH^i_{\mathfrak{A}}(\bullet) := \varinjlim_J {}^G\mathrm{Ext}^i_R(R/\mathfrak{A}, \bullet)$$

and

$${}^GD^i_{\mathfrak{A}}(\bullet) := \varinjlim_J {}^G\mathrm{Ext}^i_R(\mathfrak{A}, \bullet)$$

for $i \in \mathbb{Z}$. Thus, we get two universal δ-functors $({}^GH^i_{\mathfrak{A}})_{i\in\mathbb{Z}}$ and $({}^GD^i_{\mathfrak{A}})_{i\in\mathbb{Z}}$ from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$. For $i \in \mathbb{Z}$, the functors ${}^GH^i_{\mathfrak{A}}$ and ${}^GD^i_{\mathfrak{A}}$ respectively are called *the i-th local cohomology functor with respect to $\mathfrak{A}$* and *the i-th higher ideal transformation functor with respect to $\mathfrak{A}$.*

The functor ${}^GD^0_{\mathfrak{A}} = \varinjlim_J {}^G\mathrm{Hom}_R(\mathfrak{A}, \bullet)$ is denoted by ${}^GD_{\mathfrak{A}}$ and called *the ideal transformation functor with respect to $\mathfrak{A}$.*

(4.4.2) For every $j \in J$, there is a canonical isomorphism of functors

$${}^G\mathrm{Hom}_R(R/\mathfrak{a}_j, \bullet) \cong (0 :_\bullet \mathfrak{a}_j).$$

As $(0 :_\bullet \mathfrak{a}_j)$ is a subfunctor of $\mathrm{Id}_{\mathsf{GrMod}^G(R)}$ for every $j \in J$, taking inductive limits yields a monomorphism of functors ${}^GH^0_{\mathfrak{A}} \rightarrowtail \varinjlim_J I$, where I denotes the constant functor from J to $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$ with value $\mathrm{Id}_{\mathsf{GrMod}^G(R)}$.

As J is connected[4], it holds $\varinjlim_J I = \mathrm{Id}_{\mathsf{GrMod}^G(R)}$, and this implies

$$\varinjlim_J (0 :_\bullet \mathfrak{a}_j) = \bigcup_{j\in J}(0 :_\bullet \mathfrak{a}_j) = {}^G\Gamma_{\mathfrak{A}}.$$

Hence, there is a canonical isomorphism of functors ${}^GH^0_{\mathfrak{A}} \cong {}^G\Gamma_{\mathfrak{A}}$, by means of which we identify these functors. Since ${}^G\Gamma_{\mathfrak{A}}$ is obviously left exact, it follows that $({}^GH^i_{\mathfrak{A}})_{i\in\mathbb{Z}}$ is the right derived cohomological functor of ${}^G\Gamma_{\mathfrak{A}}$.

(4.4.3) Let $\mathfrak{a} \subseteq R$ be a G-graded ideal. We consider the multiplicative projective system $\mathfrak{A} = (\mathfrak{a}^n)_{n\in\mathbb{N}_0}$ of G-graded ideals of R over $\mathbb{N}_0$ (see 3.5.4). Then, in the notations in 4.4.1 (and in the notations to be introduced below), we write $\mathfrak{a}$ instead of $\mathfrak{A}$, getting in particular functors ${}^GH^i_{\mathfrak{a}}$, ${}^GD^i_{\mathfrak{a}}$ and ${}^GD_{\mathfrak{a}}$ called *the i-th local cohomology functor with respect to* $\mathfrak{a}$, *the i-th higher ideal transformation functor with respect to* $\mathfrak{a}$, and *the ideal transformation functor with respect to* $\mathfrak{a}$. Thus, ${}^GH^i_{\mathfrak{a}}$ is the i-th right derived functor of ${}^G\Gamma_{\mathfrak{a}}$ for every $i \in \mathbb{Z}$ by 4.4.2.

What we did in 4.3 on graded Ext functors now leads quickly to conditions for local cohomology and higher ideal transformations to commute with coarsening functors.

(4.4.4) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in* Ab.

a) The morphism of functors

$$h_{R/\mathfrak{A},\psi} : {}^GH^0_{\mathfrak{A}}(\bullet)_{[\psi]} \to {}^HH^0_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})$$

is an isomorphism.

b) If $\mathfrak{a}_j$ has a finitely generated projective resolution for every $j \in J$, then the morphism of δ-functors

$$(h^i_{R/\mathfrak{A},\psi})_{i\in\mathbb{Z}} : \big({}^GH^i_{\mathfrak{A}}(\bullet)_{[\psi]}\big)_{i\in\mathbb{Z}} \to \big({}^HH^i_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})\big)_{i\in\mathbb{Z}}$$

is an isomorphism.

Proof. Clear from 4.3.10. □

(4.4.5) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in* Ab.

a) If $\mathfrak{a}_j$ is finitely generated for every $j \in J$, then the morphism of functors

$$h_{\mathfrak{A},\psi} : {}^GD_{\mathfrak{A}}(\bullet)_{[\psi]} \to {}^HD_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})$$

is an isomorphism.

b) If $\mathfrak{a}_j$ has a finitely generated projective resolution for every $j \in J$, then the morphism of δ-functors

$$(h^i_{\mathfrak{A},\psi})_{i\in\mathbb{Z}} : \big({}^GD^i_{\mathfrak{A}}(\bullet)_{[\psi]}\big)_{i\in\mathbb{Z}} \to \big({}^HD^i_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})\big)_{i\in\mathbb{Z}}$$

is an isomorphism.

Proof. Clear from 4.3.10. □

[4] An ordered set $(E, \leq)$ is called *connected* if for all $x, y \in E$ there is a finite sequence $(z_i)_{i=0}^n$ in E with $z_0 = x$ and $z_n = y$ such that $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for every $i \in [0, n-1]$.

The next step is to make a connection between local cohomology functors and higher ideal transformation functors as in [2, 2.2.4]. This result is one of the main ingredients of the proof of the toric Serre-Grothendieck correspondence (see IV.4.2.9).

(4.4.6) The $\mathfrak{A}$-torsion functor ${}^G\Gamma_{\mathfrak{A}}$ is a subfunctor of $\mathrm{Id}_{\mathsf{GrMod}^G(R)}$, and we denote by $\xi_{\mathfrak{A}} : {}^G\Gamma_{\mathfrak{A}} \rightarrowtail \mathrm{Id}_{\mathsf{GrMod}^G(R)}$ the canonical injection. Furthermore, we denote by $\overline{\xi}_{\mathfrak{A}} : \mathrm{Id}_{\mathsf{GrMod}^G(R)} \to /{}^G\Gamma_{\mathfrak{A}}$ the cokernel of $\xi_{\mathfrak{A}}$, that is, the morphism of functors mapping a G-graded R-module M onto the canonical epimorphism $\overline{\xi}_{\mathfrak{A}}(M) : M \twoheadrightarrow M/{}^G\Gamma_{\mathfrak{A}}(M)$ in $\mathsf{GrMod}^G(R)$. Moreover, we denote by $\mathbb{X}_{\mathfrak{A}}$ the exact sequence of functors

$$0 \longrightarrow {}^G\Gamma_{\mathfrak{A}} \xrightarrow{\xi_{\mathfrak{A}}} \mathrm{Id}_{\mathsf{GrMod}^G(R)} \xrightarrow{\overline{\xi}_{\mathfrak{A}}} /{}^G\Gamma_{\mathfrak{A}} \longrightarrow 0.$$

(4.4.7) Proposition *a) There is an exact sequence of functors*

$$0 \longrightarrow {}^G\Gamma_{\mathfrak{A}} \xrightarrow{\xi_{\mathfrak{A}}} \mathrm{Id}_{\mathsf{GrMod}^G(R)} \xrightarrow{\eta_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \xrightarrow{\zeta_{\mathfrak{A}}} {}^G H^1_{\mathfrak{A}} \longrightarrow 0.$$

b) There is a unique morphism of δ-functors

$$(\zeta^i_{\mathfrak{A}})_{i\in\mathbb{Z}} : \left({}^G D^i_{\mathfrak{A}}\right)_{i\in\mathbb{Z}} \to \left({}^G H^{i+1}_{\mathfrak{A}}\right)_{i\in\mathbb{Z}}$$

such that $\zeta^0_{\mathfrak{A}} = \zeta_{\mathfrak{A}}$, and $\zeta^i_{\mathfrak{A}}$ is an isomorphism of functors for every $i \in \mathbb{N}$.

Proof. For every $j \in J$, we have the exact sequence

$$\mathbb{S}_j : 0 \longrightarrow \mathfrak{a}_j \longrightarrow R \longrightarrow R/\mathfrak{a}_j \longrightarrow 0$$

in $\mathsf{GrMod}^G(R)$. By 2.3.3 it holds ${}^G\mathrm{Ext}^0_R(R, \bullet) = \mathrm{Id}_{\mathsf{GrMod}^G(R)}$. Hence, if M is a G-graded R-module, the cohomology sequence of ${}^G\mathrm{Hom}_R(\bullet, M)$ associated with $\mathbb{S}_j$ yields an exact sequence of G-graded R-modules

$$0 \longrightarrow {}^G\mathrm{Hom}_R(R/\mathfrak{a}_j, M) \longrightarrow M \longrightarrow$$

$${}^G\mathrm{Hom}_R(\mathfrak{a}_j, M) \xrightarrow{\zeta_j(M)} {}^G\mathrm{Ext}^1_R(R/\mathfrak{a}_j, M) \longrightarrow 0$$

and for every $i \in \mathbb{N}$ an isomorphism

$$\overline{\zeta}^i_j(M) : {}^G\mathrm{Ext}^i_R(\mathfrak{a}_j, M) \xrightarrow{\cong} {}^G\mathrm{Ext}^{i+1}_R(R/\mathfrak{a}_j, M)$$

in $\mathsf{GrMod}^G(R)$. These sequences and isomorphisms being natural in M, we get an exact sequence

$$0 \longrightarrow {}^G\mathrm{Hom}_R(R/\mathfrak{a}_j, \bullet) \longrightarrow M \longrightarrow {}^G\mathrm{Hom}_R(\mathfrak{a}_j, \bullet) \xrightarrow{\zeta_j} {}^G\mathrm{Ext}^1_R(R/\mathfrak{a}_j, \bullet) \longrightarrow 0$$

and for every $i \in \mathbb{N}_0$ an isomorphism

$$\overline{\zeta}^i_j : {}^G\mathrm{Ext}^i_R(\mathfrak{a}_j, \bullet) \xrightarrow{\cong} {}^G\mathrm{Ext}^{i+1}_R(R/\mathfrak{a}_j, \bullet)$$

in $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$. These sequences and isomorphisms are readily checked to be also natural in j. Hence, by taking inductive limits we obtain an exact sequence of functors as in a) as well as for every $i \in \mathbb{N}$ an isomorphism of functors $\overline{\zeta}^i_{\mathfrak{A}} : {}^G D^i_{\mathfrak{A}} \xrightarrow{\cong} {}^G H^{i+1}_{\mathfrak{A}}$.

Clearly, $({}^G H^{i+1}_{\mathfrak{A}})_{i\in\mathbb{Z}}$ is a δ-functor from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$. As $({}^G D^i_{\mathfrak{A}})_{i\in\mathbb{Z}}$ is a universal δ-functor, there exists a unique morphism of δ-functors $(\zeta^i_{\mathfrak{A}})_{i\in\mathbb{Z}} : ({}^G D^i_{\mathfrak{A}})_{i\in\mathbb{Z}} \to ({}^G H^{i+1}_{\mathfrak{A}})_{i\in\mathbb{Z}}$ such that $\zeta^0_{\mathfrak{A}} = \zeta_{\mathfrak{A}}$. It remains to show that $\zeta^i_{\mathfrak{A}}$ is an isomorphism for $i \in \mathbb{N}$.

Let $\mathbb{T} : 0 \to L \to M \to N \to 0$ be an exact sequence in $\mathsf{GrMod}^G(R)$. For every $j \in J$, the diagram

$$\begin{tikzcd}
{}^G\mathrm{Ext}^i_R(\mathfrak{a}_j, N) \arrow[r, "\overline{\zeta}^i_j(N)"] \arrow[d] & {}^G\mathrm{Ext}^{i+1}_R(R/\mathfrak{a}_j, N) \arrow[d] \\
{}^G\mathrm{Ext}^{i+1}_R(\mathfrak{a}_j, L) \arrow[r, "\overline{\zeta}^{i+1}_j(L)"] & {}^G\mathrm{Ext}^{i+2}_R(R/\mathfrak{a}_j, L)
\end{tikzcd}$$

in $\mathsf{GrMod}^G(R)$, where the unmarked morphisms are the connecting morphisms associated with $\mathbb{T}$, is anticommutative by [3, V.4.1]. As this diagram is natural in j, taking inductive limits yields an anticommutative diagram

$$\begin{tikzcd}
{}^G D^i_{\mathfrak{A}}(N) \arrow[r, "\overline{\zeta}^i_{\mathfrak{A}}(N)"] \arrow[d] & {}^G H^{i+1}_{\mathfrak{A}}(N) \arrow[d] \\
{}^G D^{i+1}_{\mathfrak{A}}(L) \arrow[r, "\overline{\zeta}^{i+1}_{\mathfrak{A}}(L)"] & {}^G H^{i+2}_{\mathfrak{A}}(L)
\end{tikzcd}$$

in $\mathsf{GrMod}^G(R)$, where the unmarked morphisms are again the connecting morphisms associated with $\mathbb{T}$. This implies that $\zeta^i_{\mathfrak{A}} = (-1)^i \overline{\zeta}^i_{\mathfrak{A}}$ for every $i \in \mathbb{Z}$, and from this follows claim b). □

(4.4.8) The exact sequence of functors

$$0 \longrightarrow {}^G\Gamma_{\mathfrak{A}} \xrightarrow{\xi_{\mathfrak{A}}} \mathrm{Id}_{\mathsf{GrMod}^G(R)} \xrightarrow{\eta_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \xrightarrow{\zeta_{\mathfrak{A}}} {}^G H^1_{\mathfrak{A}} \longrightarrow 0$$

in 4.4.7 a) is denoted by $\mathbb{Y}_{\mathfrak{A}}$, and the induced exact sequence of functors

$$0 \longrightarrow /{}^G\Gamma_{\mathfrak{A}} \xrightarrow{\overline{\eta}_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \xrightarrow{\zeta_{\mathfrak{A}}} {}^G H^1_{\mathfrak{A}} \longrightarrow 0$$

is denoted by $\overline{\mathbb{Y}}_{\mathfrak{A}}$. Clearly, it holds $\eta_{\mathfrak{A}} = \overline{\eta}_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}}$.

Next, we generalise some more basic results on local cohomology to the graded situation; for the ungraded statements with Noetherian hypothesis see [2, 2.1.3; 2.1.7; 2.2.8].

(4.4.9) Proposition *a) It holds* $\xi_{\mathfrak{a}} \circ {}^G\Gamma_{\mathfrak{A}} = {}^G\Gamma_{\mathfrak{A}} \circ \xi_{\mathfrak{A}} = \mathrm{Id}_{{}^G\Gamma_{\mathfrak{A}}}$.

b) For every $i \in \mathbb{Z}$ *it holds* ${}^G\Gamma_{\mathfrak{A}} \circ {}^G H^i_{\mathfrak{A}} = {}^G H^i_{\mathfrak{A}}$.

Proof. By 3.6.1 it holds ${}^G\Gamma_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} = {}^G\Gamma_{\mathfrak{A}}$, and this implies claim a). Now, let $i \in \mathbb{Z}$, let M be a G-graded R-module, and let (I, d) be an injective resolution of M. Then, 4.4.2 implies that ${}^G H^i_{\mathfrak{A}}(M) = H^i({}^G\Gamma_{\mathfrak{A}}(I))$ is a quotient of $\mathrm{Ker}({}^G\Gamma_{\mathfrak{A}}(d^n))$. As ${}^G\Gamma_{\mathfrak{A}}$ is left exact, it follows from 3.6.1 that ${}^G\Gamma_{\mathfrak{A}}({}^G H^i_{\mathfrak{A}}(M)) = {}^G H^i_{\mathfrak{A}}(M)$. From this we get claim b). □

(4.4.10) Proposition *Suppose that R has the ITR-property with respect to $\mathfrak{A}$, and let $i \in \mathbb{N}$. Then it holds ${}^G H^i_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} = 0$, and the morphism of functors ${}^G H^i_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}} : {}^G H^i_{\mathfrak{A}} \to {}^G H^i_{\mathfrak{A}} \circ /{}^G\Gamma_{\mathfrak{A}}$ is an isomorphism.*

Proof. Let M be a G-graded R-module. By our hypothesis on R, there is an injective resolution I of ${}^G\Gamma_{\mathfrak{A}}(M)$ such that ${}^G\Gamma_{\mathfrak{A}}(I) = I$. In particular, the cocomplex ${}^G\Gamma_{\mathfrak{A}}(I)$ is exact at every place $i \in \mathbb{N}$, and from this follows the first claim. This implies moreover that the cohomology sequence of ${}^G\Gamma_{\mathfrak{A}}$ associated with $\mathbb{X}_{\mathfrak{A}}$ yields for every $i \in \mathbb{N}$ an exact sequence of functors

$$0 = {}^G H^i_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} \xrightarrow{{}^G H^i_{\mathfrak{A}} \circ \xi_{\mathfrak{A}}} {}^G H^i_{\mathfrak{A}} \xrightarrow{{}^G H^i_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}}} {}^G H^i_{\mathfrak{A}} \circ /{}^G\Gamma_{\mathfrak{A}} \longrightarrow {}^G H^{i+1}_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} = 0,$$

and from this follows the second claim. □

(4.4.11) Proposition *Suppose that R has the ITR-property with respect to $\mathfrak{A}$.*

a) It holds ${}^G D_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} = 0$.

b) The morphism of functors ${}^G D_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}} : {}^G D_{\mathfrak{A}} \to {}^G D_{\mathfrak{A}} \circ /{}^G\Gamma_{\mathfrak{A}}$ is an isomorphism.

c) It holds ${}^G D_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} = \eta_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}}$, and this morphism of functors is an isomorphism.

d) It holds ${}^G\Gamma_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} = {}^G H^1_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} = 0$.

e) The morphism of functors ${}^G H^i_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} : {}^G H^i_{\mathfrak{A}} \to {}^G H^i_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}}$ is an isomorphism for every $i > 1$.

Proof. a) From 4.4.10 it follows that the exact sequence of functors $\mathbb{Y}_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}}$ yields an exact sequence

$${}^G\Gamma_{\mathfrak{A}} \xrightarrow{\cong} {}^G\Gamma_{\mathfrak{A}} \xrightarrow{\eta_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} \xrightarrow{\zeta_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}}} {}^G H^1_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} = 0,$$

and this implies the claim.

b) By a), the cohomology sequence of ${}^G D_{\mathfrak{A}}$ associated with $\mathbb{X}_{\mathfrak{A}}$ yields an exact sequence of functors

$$0 = {}^G D_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} \xrightarrow{{}^G D_{\mathfrak{A}} \circ \xi_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \xrightarrow{{}^G D_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}}} {}^G D^1_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}}.$$

From 4.4.7 b) and 4.4.10 it follows ${}^G D^1_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} \cong {}^G H^2_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} = 0$, and this implies the claim.

c) Let M be a G-graded R-module, and for every G-graded R-module N and every $j \in J$ let $\iota_{N,j}$ denote the canonical injection from ${}^G\mathrm{Hom}_R(\mathfrak{a}_j, N)$ into $\varinjlim_J {}^G\mathrm{Hom}_R(\mathfrak{A}, N)$. For $x \in M$ it holds $\eta_{\mathfrak{A}}(M)(x) = \iota_{M,j}(y \mapsto yx)$ for every $j \in J$. Hence, for $\varphi \in {}^G D_{\mathfrak{A}}(M)$ we have

$$\eta_{\mathfrak{A}}({}^G D_{\mathfrak{A}}(M))(\varphi) = \iota_{{}^G D_{\mathfrak{A}}(M),j}(y \mapsto y\varphi)$$

for every $j \in J$. Moreover, for $j \in J$ and $\varphi \in {}^G\mathrm{Hom}_R(\mathfrak{a}_j, M)$ it holds

$${}^G D_{\mathfrak{A}}(\eta_{\mathfrak{A}}(M))(\iota_{M,j}(\varphi)) = \iota_{{}^G D_{\mathfrak{A}}(M),j}(\eta_{\mathfrak{A}}(M) \circ \varphi).$$

Now, let $j \in J$ and let $\varphi \in {}^G\mathrm{Hom}_R(\mathfrak{a}_j, M)$. Then, for every $y \in \mathfrak{a}_j$ we have

$$\eta_{\mathfrak{A}}(M)(\varphi(y)) = (z \mapsto z\varphi(y)) = (z \mapsto y\varphi(z)) = y\varphi,$$

hence

$$\eta_{\mathfrak{A}}({}^G D_{\mathfrak{A}}(M))(\iota_{M,j}(\varphi)) = \iota_{{}^G D_{\mathfrak{A}}(M),j}(y \mapsto y\varphi) =$$
$$\iota_{{}^G D_{\mathfrak{A}}(M),j}(\eta_{\mathfrak{A}}(M) \circ \varphi) = {}^G D_{\mathfrak{A}}(\eta_{\mathfrak{A}}(M))(\iota_{M,j}(\varphi))$$

and therefore $\eta_{\mathfrak{A}}({}^G D_{\mathfrak{A}}(M)) = {}^G D_{\mathfrak{A}}(\eta_{\mathfrak{A}}(M))$. Thus, we get $\eta_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} = {}^G D_{\mathfrak{A}} \circ \eta_{\mathfrak{A}}$. By a) and 4.4.9 b) and as ${}^G D_{\mathfrak{A}}$ is left exact, the sequence ${}^G D_{\mathfrak{A}} \circ \overline{\mathbb{Y}}_{\mathfrak{A}}$ yields an exact sequence

$$0 \longrightarrow {}^G D_{\mathfrak{A}} \circ /{}^G\Gamma_{\mathfrak{A}} \xrightarrow{{}^G D_{\mathfrak{A}} \circ \overline{\eta}_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} \xrightarrow{{}^G D_{\mathfrak{A}} \circ \zeta_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \circ {}^G H^1_{\mathfrak{A}} = 0.$$

Hence, ${}^G D_{\mathfrak{A}} \circ \overline{\eta}_{\mathfrak{A}}$ is an isomorphism. Therefore, it follows from b) and 4.4.8 that

$${}^G D_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} = {}^G D_{\mathfrak{A}} \circ (\overline{\eta}_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}}) = ({}^G D_{\mathfrak{A}} \circ \overline{\eta}_{\mathfrak{A}}) \circ ({}^G D_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}})$$

is an isomorphism, too.

d) The exact sequence of functors $\mathbb{Y}_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}}$ has the form

$$0 \to {}^G\Gamma_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} \xrightarrow{\xi_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \xrightarrow{\eta_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}}} {}^G D_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} \xrightarrow{\zeta^0_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}}} {}^G H^1_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} \to 0.$$

By c), the morphism $\eta_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}}$ is an isomorphism, and this yields the claim.

e) Let $i > 1$. Then, the cohomology sequence of ${}^G\Gamma_{\mathfrak{A}}$ associated with $\overline{\mathbb{Y}}_{\mathfrak{A}}$ yields an exact sequence

$${}^G H^{i-1}_{\mathfrak{A}} \circ {}^G H^1_{\mathfrak{A}} \to {}^G H^i_{\mathfrak{A}} \circ (\bullet/{}^G\Gamma_{\mathfrak{A}}(\bullet)) \xrightarrow{{}^G H^i_{\mathfrak{A}} \circ \overline{\eta}_{\mathfrak{A}}} {}^G H^i_{\mathfrak{A}} \circ {}^G D_{\mathfrak{A}} \xrightarrow{{}^G H^i_{\mathfrak{A}} \circ \zeta^0_{\mathfrak{A}}} {}^G H^i_{\mathfrak{A}} \circ {}^G H^1_{\mathfrak{A}}.$$

For $j \in \{i-1, i\}$, it holds

$${}^G H^j_{\mathfrak{A}} \circ {}^G H^1_{\mathfrak{A}} = {}^G H^j_{\mathfrak{A}} \circ {}^G\Gamma_{\mathfrak{A}} \circ {}^G H^1_{\mathfrak{A}} = 0$$

by 4.4.9 b) and 4.4.10, and therefore ${}^G H^i_{\mathfrak{A}} \circ \overline{\eta}_{\mathfrak{A}}$ is an isomorphism of functors. As ${}^G H^i_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}}$ is an isomorphism of functors by 4.4.10, it follows from 4.4.8 that ${}^G H^i_{\mathfrak{A}} \circ \eta_{\mathfrak{A}} = ({}^G H^i_{\mathfrak{A}} \circ \overline{\eta}_{\mathfrak{A}}) \circ ({}^G H^i_{\mathfrak{A}} \circ \overline{\xi}_{\mathfrak{A}})$ is an isomorphism of functors, too. □

The rest of this section is devoted to a graded version of a certain characterisation of ideal transformation (see [2, 2.2.11–13]. This is the second main ingredient of the toric Serre-Grothendieck correspondence (see IV.4.2.9).

(4.4.12) Lemma *Let* C *and* D *be Abelian categories, let* $F : \mathsf{C} \to \mathsf{C}$ *be a functor, and let* $(T^i)_{i \in \mathbb{Z}}$ *be a* δ*-functor from* C *to* D *such that* $T^0 \circ F = T^1 \circ F = 0$. *If* $a : A \to B$ *is a morphism in* C *such that* $F(\mathrm{Ker}(a)) = \mathrm{Ker}(a)$ *and* $F(\mathrm{Coker}(a)) = \mathrm{Coker}(a)$, *then* $T^0(a)$ *is an isomorphism.*

Proof. Let $a = a'' \circ a'$ be the canonical factorisation of a over its image. Then, we have exact sequences

$$0 = T^0(F(\mathrm{Ker}(a)) \longrightarrow T^0(A) \xrightarrow{T^0(a')} T^0(\mathrm{Im}(a)) \longrightarrow T^1(F(\mathrm{Ker}(a))) = 0$$

and

$$0 \longrightarrow T^0(\mathrm{Im}(a)) \xrightarrow{T^0(a'')} T^0(B) \longrightarrow T^0(F(\mathrm{Coker}(a))) = 0.$$

So, $T^0(a')$ and $T^0(a'')$ are isomorphisms, and thus $T^0(a) = T^0(a'') \circ T^0(a')$ is an isomorphism, too. □

(4.4.13) Proposition *Let C be a category, let $T : \mathsf{C} \to \mathsf{C}$ be a functor, and let $\eta : \mathrm{Id}_{\mathsf{C}} \to T$ be a morphism of functors such that $\eta \circ T$ is an isomorphism and that $\eta \circ T = T \circ \eta$. Moreover, let $a : A \to B$ and $c : A \to C$ be morphisms in C such that $T(a)$ is an isomorphism. Then:*

a) There is a unique morphism $b : B \to T(C)$ in C with $b \circ a = \eta(C) \circ c$, and it holds

$$b = T(c) \circ T(a)^{-1} \circ \eta(B).$$

b) If c and $\eta(B)$ are isomorphisms, then so is b.

Proof. Setting $b := T(c) \circ T(a)^{-1} \circ \eta(B)$ we get

$$b \circ a = T(c) \circ T(a)^{-1} \circ \eta(B) \circ a = T(c) \circ T(a)^{-1} \circ T(a) \circ \eta(A) =$$

$$T(c) \circ \eta(A) = \eta(C) \circ c$$

and hence the existence of b. To show uniqueness, let $b' : B \to T(c)$ be a further morphism in C with $b' \circ a = \eta(C) \circ c$. Then we get $T(b') \circ T(a) = T(\eta(C)) \circ T(c)$, hence $T(b') = T(\eta(C)) \circ T(c) \circ T(a)^{-1}$, and we also get $b' = \eta(T(C))^{-1} \circ T(b') \circ \eta(B)$. Therefore, it holds

$$b' = T(\eta(C))^{-1} \circ T(b') \circ \eta(B) = T(c) \circ T(a)^{-1} \circ \eta(B) = b.$$

The second statement is clear. □

(4.4.14) Proposition *Let C be a category, let $T, S : \mathsf{C} \to \mathsf{C}$ be functors, and let $\eta : \mathrm{Id}_{\mathsf{C}} \to T$ and $\varepsilon : \mathrm{Id}_{\mathsf{C}} \to S$ be morphisms of functors such that $\eta \circ T$ and $T \circ \varepsilon$ are isomorphisms and that $\eta \circ T = T \circ \eta$. Then:*

a) There is a unique morphism of functors $\varepsilon' : S \to T$ such that $\varepsilon' \circ \varepsilon = \eta$, and it holds

$$\varepsilon' = (T \circ \varepsilon)^{-1} \circ (\eta \circ S).$$

b) ε' is a monomorphism, epimorphism, or isomorphism respectively, if and only if $\eta \circ S$ has the same property.

Proof. For every $A \in \mathrm{Ob}(\mathsf{C})$, application of 4.4.13 a) with $c = \mathrm{Id}_A$ and $a = \varepsilon(A)$ yields a unique morphism $\varepsilon'(A) : S(A) \to T(A)$ in C such that $\varepsilon'(A) \circ \varepsilon(A) = \eta(A)$, and it holds

$$\varepsilon'(A) = T(\varepsilon(A))^{-1} \circ \eta(S(A)).$$

Therefore, it suffices to show that these morphisms $\varepsilon'(A)$ are natural in A. So, let $a : A \to B$ be a morphism in C. This gives rise to a diagram

$$
\begin{tikzcd}
A \arrow[rr, "\eta(A)"] \arrow[dr, "\varepsilon(A)"'] \arrow[dd, "a"'] & & T(A) \arrow[dr, "T(\varepsilon(A))", "\cong"'] \arrow[dd, "T(a)" near end] & \\
& S(A) \arrow[ur, "\varepsilon'(A)"] \arrow[rr, "\eta(S(A))" near start] \arrow[dd, "S(a)"' near start] & & T(S(A)) \arrow[dd, "T(S(a))"] \\
B \arrow[rr, "\eta(B)" near end] \arrow[dr, "\varepsilon(B)"'] & & T(B) \arrow[dr, "T(\varepsilon(B))", "\cong"'] & \\
& S(B) \arrow[ur, "\varepsilon'(B)"'] \arrow[rr, "\eta(S(B))"'] & & T(S(B))
\end{tikzcd}
$$

in C. Its outer faces commute obviously, and as $T(\varepsilon(B))$ is an isomorphism it follows that the whole diagram commutes. □

(4.4.15) Corollary *Suppose that R has the ITR-property with respect to $\mathfrak{A}$, and let $a : A \to B$ and $c : A \to C$ be morphisms in $\mathsf{GrMod}^G(R)$ such that ${}^G\Gamma_{\mathfrak{A}}(\mathrm{Ker}(a)) = \mathrm{Ker}(a)$ and that ${}^G\Gamma_{\mathfrak{A}}(\mathrm{Coker}(a)) = \mathrm{Coker}(a)$. Then:*

a) ${}^GD_{\mathfrak{A}}(a)$ is an isomorphism.

b) There is a unique morphism $b : B \to {}^GD_{\mathfrak{A}}(C)$ in $\mathsf{GrMod}^G(R)$ such that $b \circ a = \eta_{\mathfrak{A}}(C) \circ c$, and it holds $b = {}^GD_{\mathfrak{A}}(c) \circ {}^GD_{\mathfrak{A}}(a)^{-1} \circ \eta_{\mathfrak{A}}(B)$.

c) If c and $\eta_{\mathfrak{A}}(B)$ are isomorphisms, then so is b.

Proof. Setting $T = {}^GD_{\mathfrak{A}}$, $F = {}^G\Gamma_{\mathfrak{A}}$ and $\eta = \eta_{\mathfrak{A}}$, the claim follows from 4.4.12 and 4.4.13 on use of 4.4.11 a), 4.4.7, 4.4.10 and 4.4.11 c). □

(4.4.16) Corollary *Suppose that R has the ITR-property with respect to $\mathfrak{A}$, let $S : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$ be a functor, and let*

$$\varepsilon : \mathrm{Id}_{\mathsf{GrMod}^G(R)} \to S$$

be a morphism of functors such that ${}^GD_{\mathfrak{A}} \circ \varepsilon$ is an isomorphism. Then:

a) There is a unique morphism of functors $\varepsilon' : S \to {}^GD_{\mathfrak{A}}$ such that $\varepsilon' \circ \varepsilon = \eta_{\mathfrak{A}}$, and it holds $\varepsilon' = ({}^GD_{\mathfrak{A}} \circ \varepsilon)^{-1} \circ (\eta_{\mathfrak{A}} \circ S)$.

b) ε' is a monomorphism, epimorphism, or isomorphism respectively, if and only if $\eta_{\mathfrak{A}} \circ S$ has the same property.

Proof. Setting $T = {}^GD_{\mathfrak{A}}$, the claim follows from 4.4.14 on use of 4.4.11 c). □

(4.4.17) Corollary *Suppose that R has the ITR-property with respect to $\mathfrak{A}$, let $S : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$ be a functor, and let*

$$\varepsilon : \mathrm{Id}_{\mathsf{GrMod}^G(R)} \to S$$

be a morphism of functors with ${}^G\Gamma_{\mathfrak{A}} \circ \mathrm{Ker}(\varepsilon) = \mathrm{Ker}(\varepsilon)$ and ${}^G\Gamma_{\mathfrak{A}} \circ \mathrm{Coker}(\varepsilon) =$ $\mathrm{Coker}(\varepsilon)$. Then:

a) ${}^G D_{\mathfrak{A}} \circ \varepsilon$ *is an isomorphism.*

b) There is a unique morphism of functors $\varepsilon' : S \to {}^G D_{\mathfrak{A}}$ *such that* $\varepsilon' \circ \varepsilon = \eta_{\mathfrak{A}}$*, and it holds* $\varepsilon' = ({}^G D_{\mathfrak{A}} \circ \varepsilon)^{-1} \circ (\eta_{\mathfrak{A}} \circ S)$*.*

c) ε' *is a monomorphism, epimorphism, or isomorphism respectively, if and only if* $\eta_{\mathfrak{A}} \circ S$ *has the same property, and this is the case if and only if* ${}^G\Gamma_{\mathfrak{A}} \circ S = 0$*,* ${}^G H^1_{\mathfrak{A}} \circ S = 0$*, or* ${}^G\Gamma_{\mathfrak{A}} \circ S = {}^G H^1_{\mathfrak{A}} \circ S = 0$ *respectively.*

Proof. Claim a) holds by 4.4.15, and hence claim b) and the first equivalence in claim c) follow from 4.4.16. The second equivalence in claim c) can be read off the exact sequence $\mathbb{Y}_{\mathfrak{A}} \circ S$. □

4.5. Čech cohomology

Let $\mathbf{a} = (a_i)_{i=1}^n$ *be a finite sequence in* R^{hom}*. By a* $\mathbb{Z}$*-quasigraduation on a* G*-graded* R*-module we always mean a* $(\mathbb{Z}, \mathrm{Id}_{\mathsf{GrMod}^G(R)})$*-quasigraduation.*

In this section we introduce graded Čech cohomology, and we investigate if it coincides with local cohomology; see [2, 5.1] for the ungraded situation with Noetherian hypotheses. We start by defining the graded Čech cocomplex functor.

(4.5.1) For $k, m \in \mathbb{N}_0$, we denote by $\mathscr{I}^k_m$ the set of all strictly increasing sequences of length k in $[1, m] \subseteq \mathbb{Z}$. For $k, s \in \mathbb{N}_0$, we denote by $\iota^s \in \mathscr{I}^k_{k+1}$ the map defined by

$$\iota^s(j) := \begin{cases} j, & \text{if } j < s; \\ j+1, & \text{if } j \geq s. \end{cases}$$

Then, for $m \in \mathbb{N}_0$ we have the map $\cdot^{\hat{s}} : \mathscr{I}^{k+1}_m \to \mathscr{I}^k_m,\ f \mapsto f^{\hat{s}} := f \circ \iota^s$, and for $t \in \mathbb{N}_0$ it holds

$$\cdot^{\hat{t}} \circ \cdot^{\hat{s}} = \begin{cases} \cdot^{\widehat{s-1}} \circ \cdot^{\hat{t}}, & \text{if } s > t; \\ \cdot^{\hat{s}} \circ \cdot^{\widehat{t+1}}, & \text{if } s \leq t. \end{cases}$$

Furthermore, for $k \in \mathbb{N}_0$ and $f \in \mathscr{I}^k_n$, the map $\mathbf{a}f := \mathbf{a} \circ f = (a_{f_i})_{i=1}^k$ is a sequence of length k in $\{a_1, \ldots, a_n\} \subseteq R^{\mathrm{hom}}$, and therefore we can define $\prod \mathbf{a}f = \prod_{i=1}^k a_{f_i} \in R^{\mathrm{hom}}$.

(4.5.2) Let M be a G-graded R-module. For $k \in \mathbb{Z}$, let

$${}^G C(\mathbf{a}, M)^k := \begin{cases} \bigoplus_{f \in \mathscr{I}^k_n} M_{\prod \mathbf{a}f}, & \text{if } k \geq 0; \\ 0, & \text{if } k < 0. \end{cases}$$

We denote by ${}^G C(\mathbf{a}, M)$ the $\mathbb{Z}$-quasigraded object in $\mathsf{GrMod}^G(R)$ with underlying G-graded R-module $\bigoplus_{k \in \mathbb{Z}} {}^G C(\mathbf{a}, M)^k$ and with $\mathbb{Z}$-quasigraduation $({}^G C(\mathbf{a}, M)^k)_{k \in \mathbb{Z}})$. We have ${}^G C(\mathbf{a}, M)^0 = M$, and for every $k \in \mathbb{Z} \setminus [0, n]$ it holds ${}^G C(\mathbf{a}, M)^k = 0$.

For $k \in \mathbb{Z} \setminus [0, n-1]$, let

$${}^G d(\mathbf{a}, M)^k : {}^G C(\mathbf{a}, M)^k \to {}^G C(\mathbf{a}, M)^{k+1}$$

be the zero morphism. For $k \in [0, n-1]$, $f \in \mathscr{I}_n^k$ and $g \in \mathscr{I}_n^{k+1}$, the map

$$^G d(\mathbf{a}, M)^k_{f,g} : M_{\prod \mathbf{a} f} \to M_{\prod \mathbf{a} g}$$

with $x \mapsto (-1)^{s-1} \frac{a_{g_s} x}{a_{g_s}}$ if $f = g^{\hat{s}}$ for some $s \in [1, k+1]$ and $x \mapsto 0$ otherwise is a morphism in $\mathsf{GrMod}^G(R)$, and $(^G d(\mathbf{a}, M)^k_{f,g})_{(f,g) \in \mathscr{I}_n^k \times \mathscr{I}_n^{k+1}}$ induces a morphism

$$^G d(\mathbf{a}, M)^k : {}^G C(\mathbf{a}, M)^k \to {}^G C(\mathbf{a}, M)^{k+1}$$

in $\mathsf{GrMod}^G(R)$. Thus, the family $(^G d(\mathbf{a}, M)^k)_{k \in \mathbb{Z}}$ induces a morphism

$$^G d(\mathbf{a}, M) : {}^G C(\mathbf{a}, M) \to {}^G C(\mathbf{a}, M)(-1)$$

of $\mathbb{Z}$-quasigraded objects in $\mathsf{GrMod}^G(R)$, and by abuse of language we denote the pair $(^G C(\mathbf{a}, M), {}^G d(\mathbf{a}, M))$ just by $^G C(\mathbf{a}, M)$.

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then it holds $^G C(\mathbf{a}, M)^k_{[\psi]} = {}^H C(\mathbf{a}, M_{[\psi]})^k$ and $^G d(\mathbf{a}, M)^k_{[\psi]} = {}^H d(\mathbf{a}, M_{[\psi]})^k$ for every $k \in \mathbb{Z}$, for ψ-coarsening commutes with modules of fractions and with direct sums by 2.5.3 and 2.1.3. Hence, the pair underlying $(^G C(\mathbf{a}, M), {}^G d(\mathbf{a}, M))$ is the Čech cocomplex of the R-module underlying M with respect to $\mathbf{a}$. This implies that $^G C(\mathbf{a}, M)$ is a cocomplex in $\mathsf{GrMod}^G(R)$, and we call it *the Čech cocomplex of M with respect to* $\mathbf{a}$.

(4.5.3) Let M and N be G-graded R-modules and let $h : M \to N$ be a morphism in $\mathsf{GrMod}^G(R)$. For $k \in \mathbb{Z} \setminus [0, n]$, let

$$^G C(\mathbf{a}, h)^k : {}^G C(\mathbf{a}, M)^k \to {}^G C(\mathbf{a}, N)^k$$

be the zero morphism. For $k \in [0, n]$ and $f \in \mathscr{I}_n^k$, the map

$$^G C(\mathbf{a}, h)^k_f := h_{\prod \mathbf{a} f} : M_{\prod \mathbf{a} f} \to N_{\prod \mathbf{a} f}$$

is a morphism in $\mathsf{GrMod}^G(R)$. For $k \in [0, n]$, the family $(^G C(\mathbf{a}, h)^k_f)_{f \in \mathscr{I}_n^k}$ induces a morphism

$$^G C(\mathbf{a}, h)^k : {}^G C(\mathbf{a}, M)^k \to {}^G C(\mathbf{a}, N)^k$$

in $\mathsf{GrMod}^G(R)$. Thus, the family $(^G C(\mathbf{a}, h)^k)_{k \in \mathbb{Z}}$ induces a morphism

$$^G C(\mathbf{a}, h) : {}^G C(\mathbf{a}, M) \to {}^G C(\mathbf{a}, M)$$

of $\mathbb{Z}$-quasigraded objects in $\mathsf{GrMod}^G(R)$.

If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then it clearly holds $^G C(\mathbf{a}, h)^k_{[\psi]} = {}^H C(\mathbf{a}, h_{[\psi]})^k$ for every $k \in \mathbb{Z}$. Hence, we see that the morphism of $\mathbb{Z}$-quasigraded R-modules underlying $^G C(\mathbf{a}, h)$ is equal to the morphism in $\mathsf{CCo}(\mathsf{Mod}(R))$ induced by h on the Čech cocomplexes. This implies that $^G C(\mathbf{a}, h)$ is a morphism in $\mathsf{CCo}(\mathsf{GrMod}^G(R))$.

(4.5.4) By 4.5.2 and 4.5.3, we have a functor

$$^G C(\mathbf{a}, \bullet) : \mathsf{GrMod}^G(R) \to \mathsf{CCo}(\mathsf{GrMod}^G(R)),$$

mapping a G-graded R-module M onto its Čech cocomplex with respect to $\mathbf{a}$. This functor is called *the Čech cocomplex functor with respect to* $\mathbf{a}$. If $\psi : G \twoheadrightarrow H$ is an epimorphism in Ab, then it holds

$$^{G}C(\mathbf{a}, \bullet)_{[\psi]} = {}^{H}C(\mathbf{a}, \bullet_{[\psi]}).$$

(4.5.5) Proposition *There is an isomorphism of functors*

$$\bullet \otimes_R {}^{G}C(\mathbf{a}, R) \xrightarrow{\cong} {}^{G}C(\mathbf{a}, \bullet).$$

Proof. Let M be a G-graded R-module. For $k \in [0, n]$ and $f \in \mathscr{I}_n^k$, the R-bilinear map

$$\bar{\varphi}_M^{k,f} : M \times R_{\prod \mathbf{a}f} \to M_{\prod \mathbf{a}f},\ (m, \tfrac{r}{(\prod \mathbf{a}f)^l}) \mapsto \tfrac{rm}{(\prod \mathbf{a}f)^l}$$

induces a morphism

$$\varphi_M^{k,f} : M \otimes_R R_{\prod \mathbf{a}f} \to M_{\prod \mathbf{a}f}$$

in $\mathsf{GrMod}^G(R)$ with $m \otimes \frac{r}{(\prod \mathbf{a}f)^l} \mapsto \frac{rm}{(\prod \mathbf{a}f)^l}$. This is an isomorphism, since the map

$$\psi_M^{k,f} : M_{\prod \mathbf{a}f} \to M \otimes_R R_{\prod \mathbf{a}f},\ \tfrac{m}{(\prod \mathbf{a}f)^l} \mapsto m \otimes \tfrac{1}{(\prod \mathbf{a}f)^l}$$

is the inverse of $\varphi_M^{k,f}$. Hence,

$$\varphi_M^k := \bigoplus_{f \in \mathscr{I}_n^k} \varphi_M^{k,f} : M \otimes_R {}^{G}C(\mathbf{a}, R)^k \to {}^{G}C(\mathbf{a}, M)^k$$

is an isomorphism in $\mathsf{GrMod}^G(R)$. For $k \in \mathbb{Z} \setminus [0, n]$, let

$$\varphi_M^k : M \otimes_R {}^{G}C(\mathbf{a}, R)^k \to {}^{G}C(\mathbf{a}, M)^k$$

be the zero morphism, and let φ_M be the morphism of $\mathbb{Z}$-quasigraded objects in $\mathsf{GrMod}^G(R)$ induced by the family $(\varphi_M^k)_{k \in \mathbb{Z}}$. Then, it is straightforward to verify that φ_M is a morphism and hence an isomorphism in $\mathsf{CCo}(\mathsf{GrMod}^G(R))$. Moreover it is readily checked that it is natural in M. $\square$

(4.5.6) If M is a flat G-graded R-module, then ${}^{G}C(\mathbf{a}, M)^k$ is flat for every $k \in \mathbb{Z}$ by 4.5.2 and 2.3.9. In particular, ${}^{G}C(\mathbf{a}, R)^k$ is flat for every $k \in \mathbb{Z}$.

Next, we look at graded Čech cohomology, and we show that it commutes with coarsening.

(4.5.7) Proposition *The sequence $(H^i({}^{G}C(\mathbf{a}, \bullet)))_{i \in \mathbb{Z}}$ is an exact δ-functor from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$.*

Proof. Let

$$\mathbb{S} : 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and

$$\mathbb{S}' : 0 \longrightarrow L' \longrightarrow M' \longrightarrow N' \longrightarrow 0$$

be short exact sequences in $\mathsf{GrMod}^G(R)$ and let $(u,v,w) : \mathbb{S} \to \mathbb{S}'$ be a morphism of short exact sequences in $\mathsf{GrMod}^G(R)$. For $k \in \mathbb{Z}$, the R-module ${}^G C(\mathbf{a}, R)^k$ is flat by 4.5.6. Hence, in $\mathsf{CCo}(\mathsf{GrMod}^G(R))$ we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L \otimes_R {}^G C(\mathbf{a},R) \arrow[r] \arrow[d] & M \otimes_R {}^G C(\mathbf{a},R) \arrow[r] \arrow[d] & N \otimes_R {}^G C(\mathbf{a},R) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L' \otimes_R {}^G C(\mathbf{a},R) \arrow[r] & M' \otimes_R {}^G C(\mathbf{a},R) \arrow[r] & N' \otimes_R {}^G C(\mathbf{a},R) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Each of its rows corresponds to a family of connecting morphisms, and together with 4.5.5 their naturality yields the claim. □

(4.5.8) The δ-functor $(H^i({}^G C(\mathbf{a}, \bullet)))_{i\in\mathbb{Z}}$ is denoted by $({}^G H^i(\mathbf{a}, \bullet))_{i\in\mathbb{Z}}$ and called *the Čech cohomology functor with respect to* $\mathbf{a}$.

(4.5.9) Proposition *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in* Ab. *Then, for every $i \in \mathbb{Z}$ it holds*

$${}^G H^i(\mathbf{a}, \bullet)_{[\psi]} = {}^H H^i(\mathbf{a}, \bullet_{[\psi]}).$$

PROOF. This is clear by 4.5.4 and 4.1.2. □

Now, we adapt the proof of [2, 5.1.19] to get a condition under which local cohomology and Čech cohomology coincide.

(4.5.10) Proposition *Let $b \in R^{\mathrm{hom}}$ and let $\mathbf{b} := \mathbf{a} \amalg (b)$. Then, there is an exact sequence*

$$0 \longrightarrow {}^G C(\mathbf{a}, \bullet_b)(-1) \xrightarrow{\varphi} {}^G C(\mathbf{b}, \bullet) \xrightarrow{\psi} {}^G C(\mathbf{a}, \bullet) \longrightarrow 0$$

of functors such that for every G-graded R-module M and for every $k \in \mathbb{Z}$ the sequence

$$0 \longrightarrow {}^G C(\mathbf{a}, M_b)^{k-1} \xrightarrow{\varphi_M^k} {}^G C(\mathbf{b}, M)^k \xrightarrow{\psi_M^k} {}^G C(\mathbf{a}, M)^k \longrightarrow 0$$

in $\mathsf{GrMod}^G(R)$ splits.

PROOF. Let M be a G-graded R-module. For $k \in [1, n+1]$ and $f \in \mathscr{I}_n^{k-1}$ there is a canonical isomorphism $\chi_M^{k,f} : (M_b)_{\prod \mathbf{a} f} \xrightarrow{\cong} M_{b(\prod \mathbf{a} f)}$ in $\mathsf{GrMod}^G(R)$ that is natural in M. Hence, for $k \in [1, n]$ there is an isomorphism

$$\chi_M^k := \bigoplus_{f \in \mathscr{I}_n^{k-1}} \chi_M^{k,f} : \bigoplus_{f \in \mathscr{I}_n^{k-1}} (M_b)_{\prod \mathbf{a} f} \xrightarrow{\cong} \bigoplus_{f \in \mathscr{I}_n^{k-1}} M_{b(\prod \mathbf{a} f)}$$

in $\mathsf{GrMod}^G(R)$ that is also natural in M. Furthermore, we let φ_M^k for $k \in \mathbb{Z} \setminus [1, n+1]$ and ψ_M^k for $k \in \mathbb{Z} \setminus [0, n]$ be zero morphisms, and we set $\varphi_M^{n+1} := \chi_M^{n+1, \mathrm{Id}_{[1,n]}}$ and $\psi_M^0 := \mathrm{Id}_M$. These morphisms in $\mathsf{GrMod}^G(R)$ are obviously natural in M.

Now, let $k \in [1, n]$. Then, it holds

$$\mathscr{I}_{n+1}^k = \{f \in \mathscr{I}_{n+1}^k \mid n+1 \notin f([1,k])\} \amalg \{f \in \mathscr{I}_{n+1}^k \mid n+1 = f(k)\}.$$

Hence, we have canonical bijections

$$\lambda^k : \mathscr{I}_n^k \xrightarrow{\cong} \{f \in \mathscr{I}_{n+1}^k \mid n+1 \notin f([1,k])\}$$

and

$$\kappa^k : \mathscr{I}_n^{k-1} \xrightarrow{\cong} \{f \in \mathscr{I}_{n+1}^k \mid n+1 = f(k)\}.$$

Using these we see that

$${}^G C(\mathbf{b}, M)^k = \bigoplus_{f \in \mathscr{I}_{n+1}^k} M_{\prod \mathbf{a} f} = \big(\bigoplus_{f \in \mathscr{I}_n^{k-1}} M_{\prod \mathbf{a} \kappa^k(f)} \big) \oplus \big(\bigoplus_{f \in \mathscr{I}_n^k} M_{\prod \mathbf{a} \lambda^k(f)} \big),$$

and hence we have the canonical injection

$$\iota_M^k : \bigoplus_{f \in \mathscr{I}_n^{k-1}} M_{b(\prod \mathbf{a} f)} \rightarrowtail \bigoplus_{f \in \mathscr{I}_{n+1}^k} M_{\prod \mathbf{a} f}$$

and the canonical projection

$$\psi_M^k : \bigoplus_{f \in \mathscr{I}_{n+1}^k} M_{\prod \mathbf{a} f} \to \bigoplus_{f \in \mathscr{I}_n^k} M_{\prod \mathbf{a} f}.$$

Both of these are natural in M. Moreover, we get the monomorphism $\varphi_M^k := \iota_M^k \circ \chi_M^k$ that is also natural in M. Now it is straightforward to check that the families $(\varphi_M^k)_{k \in \mathbb{Z}}$ and $(\psi_M^k)_{k \in \mathbb{Z}}$ yield the claim. $\square$

(4.5.11) Let $\mathfrak{a} \subseteq R$ be a finitely generated G-graded ideal, and let $\mathbf{a}$ be a finite, homogeneous generating system of $\mathfrak{a}$. It is easy to see that ${}^G H^0(\mathbf{a}, \bullet) = {}^G \Gamma_{\mathfrak{a}}(\bullet)$. As the local cohomology functors with respect to $\mathfrak{a}$ are the right derived functors of ${}^G\Gamma_{\mathfrak{a}}$ by 4.4.3, universality and 4.5.7 yield a unique morphism of δ-functors

$$(g_{\mathbf{a}}^i)_{i \in \mathbb{Z}} : \big({}^G H_{\mathfrak{a}}^i(\bullet)\big)_{i \in \mathbb{Z}} \to \big({}^G H^i(\mathbf{a}, \bullet)\big)_{i \in \mathbb{Z}}$$

such that $g_{\mathbf{a}}^0 = \mathrm{Id}_{{}^G\Gamma_{\mathfrak{a}}}$.

(4.5.12) Proposition *Let $\mathfrak{a} \subseteq R$ be a finitely generated G-graded ideal, let $\mathbf{a}$ be a finite, homogeneous generating system of $\mathfrak{a}$, and suppose that R has the ITI-property with respect to every finitely generated G-graded ideal of R. Then, the morphism of δ-functors*

$$(g_{\mathbf{a}}^i)_{i \in \mathbb{Z}} : \big({}^G H_{\mathfrak{a}}^i(\bullet)\big)_{i \in \mathbb{Z}} \to \big({}^G H^i(\mathbf{a}, \bullet)\big)_{i \in \mathbb{Z}}$$

is an isomorphism.

Proof. By [6, 2.2.1] and 4.5.7 it suffices to show that ${}^G H^i(\mathbf{a}, \bullet)$ is effaceable for every $i \in \mathbb{N}$. So, let I be an injective G-graded R-module. We show the claim by induction on the number n of elements in $\mathbf{a} = (a_i)_{i=1}^n$. If $n = 0$, then it holds ${}^G C(\mathbf{a}, I) = 0$ and hence ${}^G H^i(\mathbf{a}, I) = 0$ for every $i \in \mathbb{Z}$.

So, let $n > 0$ and assume the claim to be true for strictly smaller values of n. We set $\mathbf{b} := (a_i)_{i=1}^{n-1}$ and $\mathfrak{b} := \langle \mathbf{b} \rangle_R$. By 4.5.10, we have an exact sequence

$$0 \longrightarrow {}^G C(\mathbf{b}, I_{a_n})(-1) \longrightarrow {}^G C(\mathbf{a}, I) \longrightarrow {}^G C(\mathbf{b}, I) \longrightarrow 0$$

in $\mathsf{CCo}(\mathsf{GrMod}^G(R))$. Hence, for every $i \in \mathbb{Z}$ we have an exact sequence

$${}^G H^{i-1}(\mathbf{b}, I) \xrightarrow{\delta^{i-1}} {}^G H^{i-1}(\mathbf{b}, I_{a_n}) \longrightarrow {}^G H^i(\mathbf{a}, I) \longrightarrow {}^G H^i(\mathbf{b}, I)$$

in $\mathsf{GrMod}^G(R)$. Moreover, our hypothesis implies that ${}^G H^i(\mathbf{b}, I) = 0$ for $i > 0$ and that ${}^G H^{i-1}(\mathbf{b}, I) = 0$ for $i > 1$.

In particular, for $i > 1$ we have ${}^G H^{i-1}(\mathbf{b}, I_{a_n}) \cong {}^G H^i(\mathbf{a}, I)$. Now, consider the exact sequence

$$\mathbb{S} : 0 \longrightarrow {}^G\Gamma_{\langle a_n \rangle_R}(I) \longrightarrow I \longrightarrow I_{a_n} \longrightarrow 0$$

in $\mathsf{GrMod}^G(R)$. As R has the ITI-property with respect to $\langle a_n \rangle_R$, this sequence splits, and hence I_{a_n} is injective. Therefore, the induction hypothesis implies that ${}^G H^i(\mathbf{a}, I) \cong {}^G H^{i-1}(\mathbf{b}, I_{a_n}) = 0$ for every $i > 1$.

Finally, we have to consider the case $i = 1$. By 4.5.11, the morphism

$$\delta^0 : {}^G H^0(\mathbf{b}, I) \to {}^G H^0(\mathbf{b}, I_{a_n})$$

in $\mathsf{GrMod}^G(R)$ equals

$${}^G\Gamma_{\mathfrak{b}}(\eta_{a_n}) : {}^G\Gamma_{\mathfrak{b}}(I) \to {}^G\Gamma_{\mathfrak{b}}(I_{a_n}),$$

where $\eta_{a_n} : I \to I_{a_n}$ denotes the canonical morphism in $\mathsf{GrMod}^G(R)$. Therefore, it suffices to show that ${}^G\Gamma_{\mathfrak{b}}(\eta)$ is an epimorphism. From the cohomology sequence of ${}^G\Gamma_{\mathfrak{b}}$ associated with $\mathbb{S}$ we see that this is equivalent to ${}^G H^1_{\mathfrak{b}}({}^G\Gamma_{\langle a_n \rangle_R}(I)) = 0$. But, as R has the ITI-property with respect to $\langle a_n \rangle_R$, this holds by the induction hypothesis. Thus, the claim is proven. □

(4.5.13) From the proof of 4.5.12 it is seen that instead of having the ITI-property with respect to every finitely generated G-graded ideal it suffices if R has the ITI-property with respect to $\langle a_i \rangle_R$ for every $i \in [1, n]$.

Since Čech cohomology commutes with coarsening, the above result gives a condition for local cohomology to commute with coarsening, different from the one given in 4.4.4.

(4.5.14) Corollary *Let $\psi : G \twoheadrightarrow H$ be an epimorphism in* Ab*, let $\mathfrak{a} \subseteq R$ be a finitely generated G-graded ideal, and assume that R and $R_{[\psi]}$ respectively have the ITI-property with respect to every finitely generated G- and H-graded ideal. Then, the morphism of δ-functors*

$$(h^i_{R/\mathfrak{a},\psi})_{i\in\mathbb{Z}} : \left({}^G H^i_{\mathfrak{a}}(\bullet)_{[\psi]}\right)_{i\in\mathbb{Z}} \to \left({}^H H^i_{\mathfrak{a}_{[\psi]}}(\bullet_{[\psi]})\right)_{i\in\mathbb{Z}}$$

is an isomorphism.

Proof. Clear from 4.5.12 and 4.5.9 by choosing a finite, homogeneous system of generators of $\mathfrak{a}$. □

CHAPTER IV

Toric Schemes

In this last chapter we will put together what was done previously to obtain the desired theory of toric schemes.

In Section 1, toric schemes will be defined as schemes of the form $X_{\mathbb{M}}(R)$ as studied in Chapter I, where the projective system of monoids $\mathbb{M}$ is defined by a fan. So, the results from Chapter I immediately give a bunch of results on the geometry of toric schemes, depending on properties of the base ring. Moreover, we characterise properness of toric schemes. In contrast to the results mentioned above this property depends not on the base ring, but on the fan.

Section 2 contains the first part of our generalisation of Cox's work [10] on homogeneous coordinate rings of toric varieties. First, we introduce the Cox ring associated with a fan. As this is a ring furnished with a graduation by a finitely generated group, Chapter III provides tools to handle Cox rings and graded modules over them. Next, we show how Cox rings give rise to further schemes of the form $X_{\mathbb{M}}(R)$ as in Chapter I, called Cox schemes. So, with every fan Σ are associated a toric scheme, denoted by $X_\Sigma(R)$, and a Cox scheme, denoted by $Y_\Sigma(R)$, and moreover there is a canonical morphism $Y_\Sigma(R) \to X_\Sigma(R)$ that is natural in R, and at the end of Section 2 we show that this morphism is an isomorphism if and only if the fan Σ is full. Therefore, if Σ is full, Cox schemes yield another description of toric schemes. But also if Σ is not full, then $X_\Sigma(R)$ is isomorphic to a Cox scheme on use of an appropriate base change – a further reason to prefer toric schemes over toric varieties.

On use of homogeneous coordinate rings (that is, Cox rings) Cox explained in [10] how a graded module F over the Cox ring associated with a fan Σ gives rise to a quasicoherent sheaf $\widetilde{F}$ of modules on the toric variety $X_\Sigma(\mathbb{C})$. Moreover, he showed – if Σ is simplicial – that every quasicoherent sheaf arises like this, and that this correspondence induces a bijection between graded ideals of a certain restriction of the Cox ring that are saturated with respect to some irrevelvant ideal, and quasicoherent ideals of the structure sheaf on $X_\Sigma(\mathbb{C})$. Later, in [18] Mustaţă generalised the first statement to toric varieties defined by an arbitrary fan. Section 3 is devoted to a generalisation of these results to Cox schemes, and hence to toric schemes.

In Section 4 we treat the foundations of cohomology on toric schemes. More precisely, we show how sheaf cohomology on toric schemes and graded local cohomology over Cox rings are related. Our preparations in Chapter

III allow to prove a statement analogous to the Serre-Grothendieck correspondence for projective schemes.

Altogether, the foundations for the theory of toric schemes presented in this chapter will hopefully lead to a better understanding of toric schemes (and, as a special case, toric varieties) and provide a useful basis for future work.

1. Toric schemes

Let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, let N be a $\mathbb{Z}$-structure on V, and let $M := N^*$.

1.1. Toric schemes

We start right away by defining the objects of our main interest.

(1.1.1) By I.1.4.3 and I.1.4.4 there is a contravariant functor

$$T^M(\bullet) := \mathrm{Spec}(\bullet[M]) : \mathsf{Ann} \to \mathsf{Sch}$$

over Spec, mapping a ring R onto the monoid R-scheme $T^M(R)$ called *the M-torus over R.*

(1.1.2) Let Σ be an N-fan. Then, by II.4.3.6 this gives rise to an openly immersive projective system $\Sigma^\vee_M$ of submonoids of M over the lower semilattice Σ, and then the construction from I.1.4.10 and I.1.4.13 yields a contravariant functor

$$X_{\Sigma^\vee_M} : \mathsf{Ann} \to \mathsf{Sch}$$

over Spec that maps a ring R onto an R-scheme

$$t_{\Sigma^\vee_M}(R) : X_{\Sigma^\vee_M}(R) \to \mathrm{Spec}(R).$$

If $\Sigma \neq \emptyset$, then the zero cone 0 is the smallest element of Σ, and hence $X_{\Sigma^\vee_M}(R)$ is furnished with a canonical structure of $T^M(R)$-monomodule R-scheme. If $\Sigma = \emptyset$, then we have $X_{\Sigma^\vee_M}(R) = \emptyset$, and hence also in this case $X_{\Sigma^\vee_M}(R)$ is furnished with a canonical structure of $T^M(R)$-monomodule R-scheme. So, if R is a ring, then the $T^M(R)$-monomodule R-scheme $X_{\Sigma^\vee_M}(R)$ (and by abuse of language also its underlying R-scheme and its underlying scheme) is called *the toric scheme over R associated with Σ (and N).* If no confusion can arise we write X_Σ and t_Σ instead of $X_{\Sigma^\vee_M}$ and $t_{\Sigma^\vee_M}$.

If $\sigma \in \Sigma$, then $X_{\Sigma^\vee_M,\sigma}$ does depend not on Σ but only on σ and N, and if $\tau \preccurlyeq \sigma$, then $\iota_{\Sigma^\vee_M,\tau,\sigma} : X_{\Sigma^\vee_M,\tau} \to X_{\Sigma^\vee_M,\sigma}$ does also depend not on Σ but only on σ, τ and N. If no confusion can arise we denote $X_{\Sigma^\vee_M,\sigma}$ and $\iota_{\Sigma^\vee_M,\tau,\sigma}$ respectively by X_σ and $\iota_{\tau,\sigma}$, and moreover we denote $\iota_{\Sigma^\vee_M,\sigma} : X_\sigma \to X_\Sigma$ by ι_σ.

If $\sigma \in \Sigma$, then $\iota_{\tau,\sigma} : X_\tau \to X_\sigma$ for every $\tau \preccurlyeq \sigma$ and $\iota_\sigma : X_\sigma \to X_\Sigma$ are open immersion. Moreover, $(X_\sigma)_{\sigma\in\Sigma}$ is a finite affine open covering of X_Σ that has the intersection property.

Now, let R be a ring. For $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ we consider $X_\tau(R)$ as an open sub-R-scheme of $X_\sigma(R)$ by means of $\iota_{\tau,\sigma}(R)$, and for $\sigma \in \Sigma$ we consider $X_\sigma(R)$ as an open sub-R-scheme of $X_\Sigma(R)$ by means of $\iota_\sigma(R)$. Then, $(X_\sigma(R))_{\sigma\in\Sigma}$ is a finite affine open covering of $X_\Sigma(R)$ that has the intersection property, that is, for all $\sigma, \tau \in \Sigma$ it holds $X_\sigma(R) \cap X_\tau(R) = X_{\sigma\cap\tau}(R)$.

(1.1.3) Example If σ is a sharp N-polycone in V, then the morphisms $\iota_\sigma : X_\sigma \to X_{\mathrm{face}(\sigma)}$ and Id_{X_σ} are the same, and if R is a ring, then $X_\sigma(R) = X_{\mathrm{face}(\sigma)}(R)$ is called *the toric scheme over R associated with σ (and N).* In particular, $T^M(R)$ equals the toric scheme $X_0(R)$ over R associated with the zero cone 0 and N.

(1.1.4) Example Let Σ be an N-fan in V, and let R be a ring. By I.1.4.11 it holds $X_\Sigma(R) = \emptyset$ if and only if $\Sigma = \emptyset$ or $R = 0$.

(1.1.5) Example Let Σ be an N-fan in V, and let R be a ring. Then, $t_\Sigma(R) : X_\Sigma(R) \to \mathrm{Spec}(R)$ equals $\mathrm{Id}_{\mathrm{Spec}(R)}$ if and only if $R = 0$, or if $\Sigma \neq \emptyset$ and $n = 0$. Indeed, suppose that $t_\Sigma(R) = \mathrm{Id}_{\mathrm{Spec}(R)}$ and that $\Sigma \neq \emptyset$. Then, $\iota_0 : T^M(R) \to \mathrm{Spec}(R)$ is an open immersion and in particular a monomorphism, and hence the canonical injection ι from R into the Laurent algebra S in n indeterminates $(X_i)_{i\in I}$ over R is an epimorphism. Considering the morphism of R-algebras $f : S \to S$ with $f(X_i) = X_i^{-1}$ for every $i \in I$ we have $f \circ \iota = \mathrm{Id}_S \circ \iota$, hence epimorphy of ι yields $X_i = X_i^{-1}$ for every $i \in I$, and hence $R = 0$, or $I = \emptyset$ and thus $n = 0$. The converse holds obviously.

The next two examples show that affine and projective spaces are examples of toric schemes.

(1.1.6) Example Let σ be a full N-regular N-polycone in V, and let E be an N-regular N-generating set of σ, hence a $\mathbb{Z}$-basis of N. For a ring R we denote by $R[(Y_e)_{e\in E}]$ the polynomial algebra in the indeterminates $(Y_e)_{e\in E}$ over R. Since the dual basis E^* of E generates $\sigma^\vee$ we get an isomorphism $\mathbb{N}_0^{\oplus E} \xrightarrow{\cong} \sigma_M^\vee$ and hence an isomorphism

$$\alpha : X_\sigma(\bullet) \xrightarrow{\cong} \mathrm{Spec}(\bullet[(Y_e)_{e\in E}])$$

of contravariant functors from Ann to Sch over Spec. Now, let $\tau \preccurlyeq \sigma$, and let $F \subseteq E$ be such that $\tau = \mathrm{cone}(F)$. Then, it is easy to see that there is an isomorphism

$$\alpha_F : X_\tau(\bullet) \xrightarrow{\cong} \mathrm{Spec}(\bullet[(Y_e)_{e\in E}]_{\prod_{e\in E\setminus F} Y_e})$$

of contravariant functors from Ann to Sch over Spec such that the diagram

$$\begin{tikzcd}
X_\sigma(\bullet) \arrow[r, "\alpha"] & \mathrm{Spec}(\bullet[(Y_e)_{e\in E}]) \\
X_\tau(\bullet) \arrow[u, "\iota_{\tau,\sigma}"] \arrow[r, "\alpha_F"] & \mathrm{Spec}(\bullet[(Y_e)_{e\in E}]_{\prod_{e\in E\setminus F} Y_e}) \arrow[u, "\mathrm{Spec}\circ\eta_{\prod_{e\in E\setminus F} Y_e}"']
\end{tikzcd}$$

commutes. In particular, we have $X_\sigma(\bullet) \cong \mathbb{V}^n_\bullet$.

(1.1.7) Example Let E be a $\mathbb{Z}$-basis of N. We set

$$F := E \cup \{-\textstyle\sum_{e\in E} e\} \subseteq N$$

and $\sigma_e := \mathrm{cone}(F \setminus \{e\})$ for $e \in E$. Then, $\Omega := \bigcup_{e\in F} \mathrm{face}(\sigma_e)$ is a complete, N-regular N-fan in V by II.2.2.14. We denote by $R[(Z_f)_{f\in F}]$ the $\mathbb{Z}$-graded polynomial algebra in the indeterminates $(Z_f)_{f\in F}$ over R with respect to the constant map $F \to \mathbb{Z}$ with value 1.

If $e \in F$, then it is easy to see that there is an isomorphism

$$\beta_e : \bullet[(Y_f)_{f\in F\setminus\{e\}}] \xrightarrow{\cong} \bullet[(Z_f)_{f\in F}]_{(Z_e)}$$

of functors from Ann to Ann over $\mathrm{Id}_{\mathsf{Ann}}$, and if moreover $e' \in F \setminus \{e\}$, then there is an isomorphism

$$\beta_{e',e} : \bullet[(Y_f)_{f\in F\setminus\{e\}}]_{Y_{e'}} \xrightarrow{\cong} \bullet[(Z_f)_{f\in F}]_{(Z_e Z_{e'})}$$

of functors from Ann to Ann over $\mathrm{Id}_{\mathsf{Ann}}$ such that the diagram

$$\begin{tikzcd}
\bullet[(Y_f)_{f\in F\setminus\{e\}}] \arrow[r, "\beta_e"] \arrow[d, "\eta_{Y_{e'}}"'] & \bullet[(Z_f)_{f\in F}]_{(Z_e)} \arrow[d, "(\eta_{Z_e Z_{e'}})_0"] \\
\bullet[(Y_f)_{f\in F\setminus\{e\}}]_{Y_{e'}} \arrow[r, "\beta_{e',e}"] & \bullet[(Z_f)_{f\in F}]_{(Z_e Z_{e'})}
\end{tikzcd}$$

commutes. Hence, on use of 1.1.6 and [ÉGA, II.2.4.1] we get an isomorphism

$$X_\Omega(\bullet) \xrightarrow{\cong} \mathrm{Proj}(\bullet[(Z_f)_{f\in F}])$$

of contravariant functors from Ann to Sch over Spec that induces by restriction and coastriction for every $e \in F$ and for every ring R an isomorphism from $X_{\sigma_e}(R)$ onto the open sub-R-scheme $D_+(Z_e)$ of $\mathrm{Proj}(R[(Z_f)_{f\in F}])$.

In particular, it holds $X_\Omega(\bullet) \cong \mathbb{P}^n_\bullet$.

(1.1.8) Example Let $n = 1$, let Σ be an N-fan in V, and let R be a ring. If Σ is not full, then it holds either $\Sigma = \emptyset$ or $\Sigma = \{0\}$, and hence $X_\Sigma(R) = \emptyset$ or $X_\Sigma(R) = T^M(R)$ by 1.1.4 and 1.1.3. If Σ is full, then there is an $x \in N\setminus 0$ such that it holds either $\Sigma = \{0, \mathrm{cone}(x)\}$ or $\Sigma = \{0, \mathrm{cone}(x), \mathrm{cone}(-x)\}$, and hence $X_\Sigma(R) \cong \mathbb{V}^1_R$ or $X_\Sigma(R) \cong \mathbb{P}^1_R$ by 1.1.6 and 1.1.7.

The general results on base changes from Chapter I imply some sort of "universality" of toric schemes.

(1.1.9) Let Σ be an N-fan in V. If R is a ring, then $X_\Sigma : \mathsf{Ann} \to \mathsf{Sch}$ induces a contravariant functor $X_\Sigma : \mathsf{Alg}(R) \to \mathsf{Sch}_{/R}$ over Spec, and it follows from I.1.4.10 that there is a canonical isomorphism

$$X_\Sigma(R) \times_R \bullet \cong X_\Sigma(\bullet)$$

of contravariant functors from $\mathsf{Alg}(R)$ to $\mathsf{Sch}_{/R}$ over Spec. In particular, there is a canonical isomorphism

$$X_\Sigma(\mathbb{Z}) \times \bullet \cong X_\Sigma(\bullet)$$

of contravariant functors from Ann to Sch over Spec.

Now, let R be a ring. If $\mathfrak{a} \subseteq R$ is an ideal and $p : R \twoheadrightarrow R/\mathfrak{a}$ denotes the canonical epimorphism in Ann, then $X_\Sigma(p) : X_\Sigma(R/\mathfrak{a}) \to X_\Sigma(R)$ is a closed immersion by the above, and by means of this we consider $X_\Sigma(R/\mathfrak{a})$ as a closed sub-R-scheme of $X_\Sigma(R)$. If $f \in R$ and $\eta_f(R) : R \to R_f$ denotes the canonical epimorphism, then $X_\Sigma(\eta_f(R)) : X_\Sigma(R_f) \to X_\Sigma(R)$ is an open immersion by the above, and by means of this we consider $X_\Sigma(R_f)$ as an open sub-R-scheme of $X_\Sigma(R)$.

(1.1.10) Let Σ be an N-fan in V, and let $\Sigma' \subseteq \Sigma$ be a subfan. If R is a ring, then $X_{\Sigma'}(R) = \bigcup_{\sigma\in\Sigma'} X_\sigma(R) \subseteq X_\Sigma(R)$ is an open sub-R-scheme of $X_\Sigma(R)$, and hence there is a canonical monomorphism $X_{\Sigma'} \rightarrowtail X_\Sigma$.

The above base change behaviour of toric schemes gives rise to a technique of reduction to toric schemes defined by full fans. One should note that this feature is not available for toric varieties.

(1.1.11) Lemma *Let Σ be an N-fan in V, let $V' := \langle\Sigma\rangle$, let $N' := N \cap V'$, and let Σ' denote the set Σ considered as an N'-fan in V'. Then, there is an isomorphism*

$$X_\Sigma(\bullet) \cong X_{\Sigma'}(\bullet[N/N'])$$

of contravariant functors from Ann *to* Sch *over* Spec.

Proof. By 1.1.9 there are canonical isomorphisms

$$X_\Sigma(\bullet) \cong X_\Sigma(\mathbb{Z}) \times \bullet$$

and

$$X_{\Sigma'}(\bullet[N/N']) \cong X_{\Sigma'}(\mathbb{Z}) \times (\bullet[N/N']) \cong X_{\Sigma'}(\mathbb{Z}) \times (\mathbb{Z}[N/N']) \times \bullet$$

of contravariant functors over Spec, and hence it suffices to show that there is an isomorphism of schemes $X_\Sigma(\mathbb{Z}) \cong X_{\Sigma'}(\mathbb{Z}) \times (\mathbb{Z}[N/N'])$.

We set $M' := (N')^*$, and for $\sigma \in \Sigma$ we denote by σ' the set σ considered as an N'-polycone in Σ'. As the group M' is free, we can choose an isomorphism $M' \oplus N/N' \xrightarrow{\cong} M$ in Ab. If $\sigma \in \Sigma$, this induces by restriction and coastriction an isomorphism $(\sigma')^\vee_{M'} \oplus N/N' \xrightarrow{\cong} \sigma^\vee_M$ in Mon such that for every $\tau \preccurlyeq \sigma$ the diagram

$$\begin{array}{ccc} (\sigma')^\vee_{M'} \oplus N/N' & \xrightarrow{\cong} & \sigma^\vee_M \\ \downarrow & & \downarrow \\ (\tau')^\vee_{M'} \oplus N/N' & \xrightarrow{\cong} & \tau^\vee_M \end{array}$$

in Mon commutes. As the functor $\mathbb{Z}[\bullet] : \mathsf{Mon} \to \mathsf{Ann}$ commutes with coproducts by I.1.3.1, these isomorphisms induce for every $\sigma \in \Sigma$ an isomorphism

$$\mathbb{Z}[(\sigma')^{\vee}_{M'}] \otimes \mathbb{Z}[N/N'] \xrightarrow{\cong} \mathbb{Z}[\sigma^{\vee}_{M}]$$

in Ann and hence by taking spectra an isomorphism

$$X_\sigma(\mathbb{Z}) \xrightarrow{\cong} X_{\sigma'}(\mathbb{Z}) \times \mathrm{Spec}(\mathbb{Z}[N/N'])$$

in Sch such that for every $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
X_\sigma(\mathbb{Z}) \arrow[r, "\cong"] & X_{\sigma'}(\mathbb{Z}) \times \mathrm{Spec}(\mathbb{Z}[N/N']) \\
X_\tau(\mathbb{Z}) \arrow[r, "\cong"] \arrow[u, "\iota_{\tau,\sigma}(\mathbb{Z})"] & X_{\tau'}(\mathbb{Z}) \times \mathrm{Spec}(\mathbb{Z}[N/N']) \arrow[u, "\iota_{\tau',\sigma'}(\mathbb{Z}) \times \mathrm{Id}_{\mathrm{Spec}(\mathbb{Z}[N/N'])}"']
\end{tikzcd}$$

in Sch commutes. Thus, these isomorphisms can be glued to obtain an isomorphism $X_\Sigma(\mathbb{Z}) \xrightarrow{\cong} X_{\Sigma'}(\mathbb{Z}) \times \mathrm{Spec}(\mathbb{Z}[N/N'])$ in Sch as desired. □

(1.1.12) Note that the isomorphism in 1.1.11 depends on the choice of an isomorphism $M' \oplus N/N' \cong M$ in Ab and is therefore not canonical.

1.2. Basic properties of toric schemes

Let Σ be an N-fan in V, and let R be a ring.

In this section we apply the results from I.2 to describe some fundamental properties of toric schemes.

(1.2.1) Proposition *a) $X_\Sigma(R)$ is separated, quasicompact, flat and of finite presentation over R.*

b) $X_\Sigma(R)$ is faithfully flat over R if and only if $\Sigma \neq \emptyset$ or $R = 0$.

Proof. This follows from II.4.3.4, I.2.1.2, I.2.2.1, II.4.3.1, I.2.1.3 b) and I.2.2.2. □

(1.2.2) Proposition *a) $X_\Sigma(R)$ is reduced, connected or normal, respectivly, if and only if R is so, or $\Sigma = \emptyset$.*

b) $X_\Sigma(R)$ is irreducible or integral, respectively, if and only if R is so and $\Sigma \neq \emptyset$.

Proof. This follows from II.4.3.1, I.2.3.8, I.2.4.3 and I.2.5.3. □

(1.2.3) Proposition *If $\Sigma \neq \emptyset$, then the map $\mathfrak{p} \mapsto X_\Sigma(R/\mathfrak{p})$ defines a bijection from $\mathrm{Min}(R)$ onto the set of irreducible components of $X_\Sigma(R)$.*

Proof. Clear from II.4.3.1 and I.2.3.12. □

(1.2.4) Proposition *a) $X_\Sigma(R)$ is Noetherian if and only if R is Noetherian, or $\Sigma = \emptyset$.*

b) $X_\Sigma(R)$ is Artinian if and only if R is Artinian and $n = 0$, or $R = 0$, or $\Sigma = \emptyset$.

Proof. This holds by II.4.3.1, I.2.6.4 b), II.4.3.2 and I.2.6.9. □

(1.2.5) Proposition *a) If $\Sigma \neq \emptyset$, then it holds*

$$\dim(R) + n \leq \dim(X_\Sigma(R)) \leq \sum_{i=0}^{n-1} 2^i + 2^n \dim(R).$$

b) If R is Noetherian and $\Sigma \neq \emptyset$, then it holds

$$\dim(R) + n = \dim(X_\Sigma(R)).$$

c) If R is Noetherian, then $X_\Sigma(R)$ is equidimensional if and only if R is equidimensional, or $\Sigma = \emptyset$.

Proof. For $\sigma \in \Sigma$ it holds

$$\mathrm{rk}(\mathrm{Diff}(\sigma_M^\vee)) = \dim(\langle \mathrm{Diff}(\sigma_M^\vee)\rangle_\mathbb{R}) = \dim(\sigma^\vee) = n$$

by II.1.2.10. So, all the claims follow from II.4.3.1, I.2.6.7 and I.2.6.10. □

1.3. Projections of fans and properness of toric schemes

The aim of this section is to characterise toric schemes that are proper over their base ring. It will turn out that this property can be characterised by the fan alone. We start by showing how a toric scheme $X_\Sigma(R)$ can be covered by the toric schemes $X_{\Sigma/\sigma}(R)$ associated with projections of Σ along every cone σ in Σ.

(1.3.1) Let σ and τ be N-polycones in V with $\sigma \preccurlyeq \tau$. Then, $\tau_M^\vee \setminus \sigma^\perp$ is a prime monoideal of $\tau_M^\vee$. Indeed, for $u \in \tau_M^\vee \setminus \sigma^\perp$ and $v \in \tau_M^\vee$ it holds $v + u \in \tau_M^\vee$, and for $x \in \sigma$ we have $u(x) > 0$ and $v(x) \geq 0$, hence $(u+v)(x) > 0$ and therefore $u + v \in \tau_M^\vee \setminus \sigma^\perp$. This shows that $\tau_M^\vee \setminus \sigma^\perp$ is a monoideal of $\tau_M^\vee$, and since $\tau_M^\vee \cap \sigma^\perp$ is obviously a submonoid of $\tau_M^\vee$ it is prime.

Moreover, it is easy to see that there is surjective morphism

$$p_\sigma(\tau)_{M_\sigma}^\vee \to \tau_M^\vee \cap \sigma^\perp,\ u \mapsto u \circ p_\sigma \restriction_N$$

in Mon. Since p_σ induces by restriction and coastriction an epimorphism $N \twoheadrightarrow N_\sigma$ in Ab this morphism is a monomorphism and hence an isomorphism.

(1.3.2) Let σ and τ be N-polycones in V with $\sigma \preccurlyeq \tau$. Then, by 1.3.1 and I.1.3.16 we get a morphism

$$\vartheta_{\tau,\sigma} := \vartheta_{\tau_M^\vee, \tau_M^\vee \setminus \sigma^\perp} : \bullet[\tau_M^\vee] \to \bullet[\tau_M^\vee \cap \sigma^\perp]$$

of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$ such that $\vartheta_{\tau,\sigma}(R)$ is surjective and that

$$\vartheta_{\tau,\sigma}(R)(e_x) = \begin{cases} e_x, & \text{if } x \in \sigma^\perp; \\ 0, \text{if } x \notin \sigma^\perp \end{cases}$$

for every ring R and every $x \in \sigma_M^\vee$.

Moreover, the inverse of the isomorphism defined in 1.3.1 induces an isomorphism

$$\pi_{\tau,\sigma} : \bullet[\tau_M^\vee \cap \sigma^\perp] \xrightarrow{\cong} \bullet[p_\sigma(\tau)_{M_\sigma}^\vee]$$

of functors form Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$, and composition with $\vartheta_{\tau,\sigma}$ yields a morphism

$$\vartheta'_{\tau,\sigma} : \bullet[\tau_M^\vee] \to \bullet[p_\sigma(\tau)_{M_\sigma}^\vee]$$

of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$ such that $\vartheta'_{\tau,\sigma}(R)$ is surjective for every ring R.

For a further N-polycone ω in V with $\sigma \preccurlyeq \omega \preccurlyeq \tau$ it is readily checked on use of I.1.3.16 that the diagram

$$\begin{tikzcd}
\bullet[\tau_M^\vee] \arrow[r, "\vartheta'_{\tau,\sigma}"] & \bullet[p_\sigma(\tau)_{M_\sigma}^\vee] \\
\bullet[\omega_M^\vee] \arrow[u, tail] \arrow[r, "\vartheta'_{\omega,\sigma}"] & \bullet[p_\sigma(\omega)_{M_\sigma}^\vee] \arrow[u, tail]
\end{tikzcd}$$

of functors from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$, where the unmarked morphisms are induced by the canonical injections of $\tau_M^\vee$ and $\tau_M^\vee \cap \sigma^\perp$ into $\omega_M^\vee$ and $\omega_M^\vee \cap \sigma^\perp$, commutes.

Composing the above morphisms with Spec we get a closed immersion $\chi_{\tau,\sigma} : X_{\tau/\sigma} \to X_\tau$ of functors from Ann to Sch over Spec such that for every $\omega \preccurlyeq \tau$ with $\sigma \preccurlyeq \omega$ the diagram

$$\begin{tikzcd}
X_{\tau/\sigma} \arrow[r, "\chi_{\tau,\sigma}"] & X_\tau \\
X_{\omega/\sigma} \arrow[u, "\iota_{\omega/\sigma,\tau/\sigma}"] \arrow[r, "\chi_{\omega,\sigma}"] & X_\omega \arrow[u, "\iota_{\omega,\tau}"']
\end{tikzcd}$$

of functors from Ann to Sch over Spec commutes. If R is a ring and no confusion can arise, then we consider $X_{\tau/\sigma}(R)$ as a closed sub-R-scheme of $X_\tau(R)$ by means of $\chi_{\tau,\sigma}(R)$.

(1.3.3) Lemma *Let σ, τ and ω be N-polycones in V with $\sigma \preccurlyeq \omega \preccurlyeq \tau$. Then, the diagram*

$$\begin{tikzcd}
X_{\tau/\sigma} \arrow[r, "\chi_{\tau,\sigma}"] & X_\tau \\
X_{\omega/\sigma} \arrow[u, "\iota_{\omega/\sigma,\tau/\sigma}"] \arrow[r, "\chi_{\omega,\sigma}"] & X_\omega \arrow[u, "\iota_{\omega,\tau}"']
\end{tikzcd}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Sch})_{/\mathrm{Spec}}$ *is cartesian.*

Proof. By II.4.3.3 there is a $u \in \tau_M^\vee \cap \sigma^\perp$ with $\omega = \tau \cap u^\perp$ and $\omega_M^\vee = \tau_M^\vee \oplus \mathbb{N}_0(-u)$, and then I.1.2.10 implies $\omega_M^\vee \cap \sigma^\perp = (\tau_M^\vee \cap \sigma^\perp) \oplus \mathbb{N}_0(-u)$.

Now, let R be a ring. It suffices to show that the diagram

$$\begin{tikzcd} R[\tau_M^\vee \cap \sigma^\perp] \arrow[d, "\varepsilon"'] & R[\tau_M^\vee] \arrow[l, "\vartheta_{\tau,\sigma}(R)"'] \arrow[d, "R[\varepsilon_u]"] \\ R[\omega_M^\vee \cap \sigma^\perp] & R[\omega_M^\vee] \arrow[l, "\vartheta_{\omega,\sigma}(R)"'] \end{tikzcd}$$

in $\mathsf{Alg}(R)$, where ε is induced by restriction and coastriction of ε_u, is cocartesian. By I.1.3.8 we have

$$A := R[\tau_M^\vee \cap \sigma^\perp] \otimes_{R[\tau_M^\vee]} R[\omega_M^\vee] = R[\tau_M^\vee \cap \sigma^\perp] \otimes_{R[\tau_M^\vee]} R[\tau_M^\vee \oplus \mathbb{N}_0(-u)] \cong$$
$$R[\tau_M^\vee \cap \sigma^\perp] \otimes_{R[\tau_M^\vee]} (R[\tau_M^\vee] \otimes_R R[\mathbb{N}_0(-u)]) \cong R[\tau_M^\vee \cap \sigma^\perp] \otimes_R R[\mathbb{N}_0(-u)] \cong$$
$$R[(\tau_M^\vee \cap \sigma^\perp) \oplus \mathbb{N}_0(-u)] = R[\omega_M^\vee \cap \sigma^\perp]$$

and hence an isomorphism $\lambda : A \xrightarrow{\cong} R[\omega_M^\vee \cap \sigma^\perp]$. Denoting by λ_0 and λ_1 the canonical morphisms from $R[\tau_M^\vee \cap \sigma^\perp]$ and $R[\tau_M^\vee \oplus \mathbb{N}_0(-u)]$ to A it is readily checked that $\lambda \circ \lambda_0 \circ \vartheta_{\tau,\sigma}(R) = \varepsilon \circ \vartheta_{\tau,\sigma}(R)$ and $\lambda \circ \lambda_1 \circ R[\varepsilon_u] = \vartheta_{\omega,\sigma}(R) \circ R[\varepsilon_u]$. Since $\vartheta_{\tau,\sigma}(R)$ and $R[\varepsilon_u]$ are epimorphisms by 1.3.2, I.1.2.2 and I.1.3.13 this implies $\lambda \circ \lambda_0 = \varepsilon$ and $\lambda \circ \lambda_1 = \vartheta_{\omega,\sigma}(R)$, and thus the claim. $\square$

(1.3.4) Lemma *Let σ, τ and ω be N-polycones in V with $\sigma \preccurlyeq \tau$ and $\sigma \not\preccurlyeq \omega \preccurlyeq \tau$. If I denotes the initial object in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Sch})_{/\,\mathrm{Spec}}$, then the diagram*

$$\begin{tikzcd} X_{\tau/\sigma} \arrow[r, "\chi_{\tau,\sigma}"] & X_\tau \\ I \arrow[u] \arrow[r] & X_\omega \arrow[u, "\iota_{\omega,\tau}"'] \end{tikzcd}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Sch})_{/\,\mathrm{Spec}}$ is cartesian.

Proof. By II.4.3.3 there is a $u \in \tau_M^\vee \setminus \sigma^\perp$ with $\omega = \tau \cap u^\perp$ and $\omega_M^\vee = \tau_M^\vee \oplus \mathbb{N}_0(-u)$.

Now, let R be a ring. It suffices to show that the diagram

$$\begin{tikzcd} R[\tau_M^\vee \cap \sigma^\perp] \arrow[d] & R[\tau_M^\vee] \arrow[l, "\vartheta_{\tau,\sigma}(R)"'] \arrow[d, "R[\varepsilon_u]"] \\ 0 & R[\tau_M^\vee \oplus \mathbb{N}_0(-u)] \arrow[l] \end{tikzcd}$$

in $\mathsf{Alg}(R)$ is cocartesian, and for this it suffices to show that

$$A := R[\tau_M^\vee \cap \sigma^\perp] \otimes_{R[\tau_M^\vee]} R[\tau_M^\vee \oplus \mathbb{N}_0(-u)] = 0,$$

where $R[\tau_M^\vee \cap \sigma^\perp]$ is considered as an $R[\tau_M^\vee]$-algebra by means of $\vartheta_{\tau,\sigma}(R)$. By the above we have $\vartheta_{\tau,\sigma}(R)(e_u) = 0$, and $e_u \in R[\tau_M^\vee \oplus \mathbb{N}_0(-u)]$ is invertible. Therefore, for the unit of A we get

$$1 \otimes 1 = 1 \otimes (e_u e_{-u}) = e_u(1 \otimes e_{-u}) = \vartheta_{\tau,\sigma}(R)(e_u) \otimes e_{-u} = 0,$$

hence $A = 0$ and thus the claim. $\square$

(1.3.5) Proposition *Let σ be an N-polycone in V, and let R be a ring. Then, it holds $X_{\sigma/\sigma}(R) = X_\sigma(R) \setminus \bigcup_{\tau\prec\sigma} X_\tau(R)$.*

Proof. By 1.3.4 we have $X_{\sigma/\sigma}(R) \subseteq X_\sigma(R) \setminus \bigcup_{\tau\prec\sigma} X_\tau(R)$. Conversely, let $\mathfrak{p} \in X_\sigma(R) \setminus \bigcup_{\tau\prec\sigma} X_\tau(R)$. Let $u \in \sigma^\vee_M \setminus \sigma^\perp$. Then, $\tau := \sigma \cap u^\perp$ is a face of σ. If $e_u \notin \mathfrak{p}$, then we see on use of [AC, II.4.3 Proposition 13, Corollaire], II.4.3.3 and I.1.3.14 that there exists $\mathfrak{q} \in \mathrm{Spec}(R[\sigma^\vee_M]_{e_u}) = X_\tau(R)$ with $\mathfrak{p} = \iota_{\tau,\sigma}(R)(\mathfrak{q})$, and then the choice of $\mathfrak{p}$ implies $\tau = \sigma$ and hence the contradiction $u \in \sigma^\perp$. So, it holds $e_u \in \mathfrak{p}$, and from this we get $\mathrm{Ker}(\vartheta_{\sigma,\sigma}(R)) = \langle e_u \mid u \in \sigma^\vee_M \setminus \sigma^\perp\rangle \subseteq \mathfrak{p}$. But by [AC, II.4.3 Remarque] this is equivalent to $\mathfrak{p} \in X_{\sigma,\sigma}(R)$, and thus the claim is proven. □

(1.3.6) Lemma *Let $f : Y \to X$ be a morphism in* Sch*, and let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ respectively be open coverings of X and Y that have the intersection properties such that J is a segment*[1] *of I. Moreover, suppose that f induces for every $j \in J$ a closed immersion $f_j : Y_j \to X_j$, and that for all $i \in I$ and all $j \in J$ it holds*

$$f_j^{-1}(X_{\inf(i,j)}) = \begin{cases} Y_{\inf(i,j)}, & \text{if } i \in J; \\ \emptyset, & \text{if } i \notin J. \end{cases}$$

Then, f is a closed immersion.

Proof. Let $i \in I$, and let $g_i : f^{-1}(X_i) \to X_i$ denote the morphism induced by f. By [ÉGA, I.4.2.4] it suffices to show that g_i is a closed immersion. For every $j \in J$ it holds $Y_j \subseteq f^{-1}(X_j)$, and the intersection property implies $Y_j \cap f^{-1}(X_i) = Y_j \cap f^{-1}(X_{\inf(i,j)})$. Hence, from [ÉGA, 0.1.2.14] it follows

$$Y_j \cap f^{-1}(X_i) = Y_j \cap f^{-1}(X_{\inf(i,j)}) = Y_j \times_X X_{\inf(i,j)} =$$

$$Y_j \times_{X_j} X_{\inf(i,j)} = f_j^{-1}(X_{\inf(i,j)}).$$

Therefore, we get $f^{-1}(X_i) = \bigcup_{j\in J} f_j^{-1}(X_{\inf(i,j)})$. If $i \in J$, it follows

$$f^{-1}(X_i) = \bigcup_{j\in J} Y_{\inf(i,j)} = Y_i \subseteq f^{-1}(X_i),$$

and hence $g_i = f_i$ is a closed immersion. If $i \notin J$, we have $f^{-1}(X_i) = \emptyset$, and then g_i is obviously a closed immersion, too. Herewith, the claim is proven. □

(1.3.7) Proposition *Let Σ be an N-fan in V, and let $\sigma \in \Sigma$. Then, there exists a unique morphism $\chi_\sigma : X_{\Sigma/\sigma} \to X_\Sigma$ of functors from* Ann *to* Sch *over*

[1] If E is an ordered set, then a subset $F \subseteq E$ is called a *segment of* E if for every $x \in F$ it holds $E_{\leq x} \subseteq F$. Note that if E is a lower semilattice and F is a segment of E, then F is a lower subsemilattice of E, that is, for all $x, y \in F$ it holds $\inf_E(x, y) \in F$.

Spec *such that for every* $\tau \in \Sigma_\sigma$ *the diagram*

$$\begin{tikzcd} X_{\Sigma/\sigma} \arrow[r, "\chi_\sigma"] & X_\Sigma \\ X_{\tau/\sigma} \arrow[u, "\iota_{\tau/\sigma}"] \arrow[r, "\chi_{\tau,\sigma}"] & X_\tau \arrow[u, "\iota_\tau"'] \end{tikzcd}$$

commutes, and $\chi_\sigma : X_{\Sigma/\sigma} \to X_\Sigma$ *is a closed immersion.*

Proof. Existence and uniqueness of χ_σ are clear by 1.3.2. Let R be a ring, let $\tau \in \Sigma_\sigma$, and let $\omega \in \Sigma$. By 1.3.6 and 1.3.2 it suffices to show that

$$X_{\tau/\sigma}(R) \times_{X_\tau(R)} X_{\tau\cap\omega}(R) = X_{(\tau\cap\omega)/\sigma}(R)$$

if $\sigma \preccurlyeq \omega$ and that $X_{\tau/\sigma}(R) \times_{X_\tau(R)} X_{\tau\cap\omega}(R) = \emptyset$ if $\sigma \not\preccurlyeq \omega$. To do this we can assume without loss of generality that $\omega \preccurlyeq \tau$, and then the claim follows from 1.3.3 and 1.3.4. □

(1.3.8) Let Σ be an N-fan in V, let $\sigma \in \Sigma$, and let R be a ring. If no confusion can arise, then we consider $X_{\Sigma/\sigma}(R)$ as a closed sub-R-scheme of $X_\Sigma(R)$ by means of $\chi_\sigma(R)$.

(1.3.9) Corollary *Let* Σ *be an* N*-fan in* V*, let* $\sigma \in \Sigma$*, and let* R *be a ring. Then,* $(X_{\Sigma/\sigma}(R))_{\sigma\in\Sigma}$ *is a closed covering of* $X_\Sigma(R)$*, and* $(X_{\sigma/\sigma}(R))_{\sigma\in\Sigma}$ *is a locally closed covering of* $X_\Sigma(R)$*.*

Proof. By 1.3.5 we have

$$X_\Sigma(R) = \bigcup_{\sigma\in\Sigma}\big(X_\sigma(R)\setminus\bigcup_{\tau\prec\sigma} X_\tau(R)\big) = \bigcup_{\sigma\in\Sigma} X_{\sigma/\sigma}(R) \subseteq \bigcup_{\sigma\in\Sigma} X_{\Sigma/\sigma}(R) \subseteq X_\Sigma(R),$$

and hence the claim follows from 1.3.7. □

Now we attack the characterisation of properness by means of the well-known valuative criterion. As before, base change properties allow reducing to a Noetherian base ring and hence to *discrete* valuation rings.

(1.3.10) Lemma *Let* R *be a ring. Then, the following statements are equivalent:*

(i) $T^M(R)$ *is proper over* R*;*

(ii) $\mathbb{V}^n_R$ *is proper over* R*;*

(iii) $R = 0$ *or* $n = 0$*.*

Proof. On use of the valuative criterion for properness [ÉGA, II.7.3.8] and I.1.3.1 it is easily seen that (i) and (ii) respectively holds if and only if $n = 0$ or if for every R-algebra A that is a valuation ring[2] with field of fractions K it holds $A^* = K^*$ or $A = K$, respectively. But this is the case if and only if $n = 0$ or if there exists no R-algebra A that is a valuation ring.

[2] We use the terminology of [ÉGA], that is, a *valuation ring* is a valuation ring in the sense of [AC] that is not a field.

Clearly, this holds if $R=0$. Conversely, if $R\neq 0$, then there exists a maximal ideal $\mathfrak{m}$ in R by [AC, I.8.6 Théorème 1], and then the ring of fractions $(R/\mathfrak{m})[X]_{\langle X\rangle}$ of the polynomial algebra $(R/\mathfrak{m})[X]$ in one indeterminate over R is an R-algebra that is integral, local and Noetherian but not a field, and its maximal ideal is principal. Therefore, it is a (discrete) valuation ring by [AC, VI.3.6 Proposition 9], and thus the claim is proven. □

(1.3.11) Lemma *Let R be a ring, let Σ be a complete N-fan in V, let $\alpha: R\to A$ be an R-algebra such that A is a discrete valuation ring, let $\eta: A\to K$ be the canonical injection into the field of fractions of A, and let $f:\operatorname{Spec}(K)\to X_\Sigma(R)$ be a morphism in $\mathsf{Sch}_{/R}$. Then, there exists a morphism $g:\operatorname{Spec}(A)\to X_\Sigma(R)$ in $\mathsf{Sch}_{/R}$ such that $g\circ\operatorname{Spec}(\eta)=f$.*

Proof. By 1.3.9 there is a $\sigma\in\Sigma$ with $f(\operatorname{Spec}(K))\subseteq X_{\sigma/\sigma}(R)$, and hence by [ÉGA, I.4.6.2] there is a morphism $f':\operatorname{Spec}(K)\to X_{\sigma/\sigma}(R)$ in $\mathsf{Sch}_{/R}$ such that $\chi_\sigma(R)\circ\iota_{\sigma/\sigma}(R)\circ f'=f$. Then, f' corresponds to a morphism $p:R[M_\sigma]\to K$ in $\mathsf{Alg}(R)$, and by I.1.3.1 this corresponds to a morphism u in Mon from M_σ to the multiplicative monoid underlying K. Since M_σ is a group it holds $u(M_\sigma)\subseteq K^*$.

Now, as K is the field of fractions of the discrete valuation ring A, there is a valuation $v:K\to\mathbb{Z}_\infty$ of K with ring A, hence, in particular, a morphism v in Mon from the multiplicative monoid underlying K to $\mathbb{Z}_\infty$ such that $\{x\in K\mid v(x)\geq 0\}=A$ (see [AC, VI.3.6]). Then, coastriction of $v\circ u$ yields a morphism $w:M_\sigma\to\mathbb{Z}$ in Mon, that is, $w\in M_\sigma^*=N_\sigma$.

As Σ/σ is complete by II.2.2.7 c), there is a $\tau\in\Sigma_\sigma$ with $w\in\tau/\sigma$ and hence $(\tau/\sigma)^\vee_{M_\sigma}\subseteq w^\vee\cap M_\sigma$. It follows that for every $x\in(\tau/\sigma)^\vee_{M_\sigma}$ we have $v(u(x))=w(x)\geq 0$ and therefore $u(x)\in A$. Thus, there is a morphism t from $(\tau/\sigma)^\vee_{M_\sigma}$ to the multiplicative monoid underlying A such that the diagram

$$\begin{tikzcd} M_\sigma \arrow[r, "u"] & K \\ (\tau/\sigma)^\vee_{M_\sigma} \arrow[u, hook] \arrow[r, "t"] & A \arrow[u, "\eta"'] \end{tikzcd}$$

in Mon commutes. By I.1.3.1 we get a morphism $q:R[(\tau/\sigma)^\vee_{M_\sigma}]\to A$ in $\mathsf{Alg}(R)$ such that the diagram

$$\begin{tikzcd} R[M_\sigma] \arrow[r, "p"] & K \\ R[(\tau/\sigma)^\vee_{M_\sigma}] \arrow[u] \arrow[r, "q"] & A \arrow[u, "\eta"'] \end{tikzcd}$$

in $\mathsf{Alg}(R)$, where the unmarked morphism is induced by the canonical injection $(\tau/\sigma)^\vee_{M_\sigma}\hookrightarrow M_\sigma$, commutes. Taking spectra yields a morphism

$g' : \mathrm{Spec}(A) \to X_{\tau/\sigma}(R)$ in $\mathsf{Sch}_{/R}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{f'} & X_{\sigma/\sigma}(R) \\ {\scriptstyle \mathrm{Spec}(\eta)}\downarrow & & \downarrow{\scriptstyle \iota_{\sigma/\sigma,\tau/\sigma}(R)} \\ \mathrm{Spec}(A) & \xrightarrow{g'} & X_{\tau/\sigma}(R) \end{array}$$

in Sch/R commutes, and then with $g := \chi_\sigma(R) \circ \iota_{\tau/\sigma}(R) \circ g'$ the claim holds. □

(1.3.12) Theorem *Let Σ be an N-fan in V, and let R be a ring. Then, the following statements are equivalent:*

(i) $X_\Sigma(R)$ is proper over R;

(ii) Σ is complete, or $\Sigma = \emptyset$, or $R = 0$.

Proof. If $\Sigma = \emptyset$ or $R = 0$ this is clear. So, let $\Sigma \neq \emptyset$ and $R \neq 0$. First, suppose that $X_\Sigma(R)$ is proper over R. We show by induction on n that Σ is complete. If $n = 0$, then $\Sigma = \{0\}$ is complete. If $n = 1$ and Σ is not complete, then it holds $X_\Sigma(R) = T^M(R)$ or $X_\Sigma(R) \cong \mathbb{V}^1_R$ by 1.1.8, and then 1.3.10 yields a contradiction. So, let $n > 1$, and suppose the claim to be true for strictly smaller values of n. If $\Sigma_1 = \emptyset$, then we have $\Sigma = \{0\}$, hence $X_\Sigma(R) = T^M(R)$, and therefore 1.3.10 yields again a contradiction. Thus, it holds $\Sigma_1 \neq \emptyset$, and then by II.2.3.12 it suffices to show that Σ/ρ is complete for every $\rho \in \Sigma_1$. So, let $\rho \in \Sigma_1$. By 1.3.7 we have a closed immersion $X_{\Sigma/\rho}(R) \to X_\Sigma(R)$, and since closed immersions and compositions of proper morphisms are proper by [ÉGA, II.5.4.2] it follows that $X_{\Sigma/\rho}(R)$ is proper over R and hence Σ/ρ is complete by our hypothesis.

Conversely, suppose that Σ is complete. As $\mathbb{Z}$ is Noetherian and $t_\Sigma(\mathbb{Z})$ is quasicompact and of finite type by 1.2.1 a), the valuative criterion for properness [ÉGA, II.7.3.8] and 1.3.11 imply that $t_\Sigma(\mathbb{Z})$ is proper. Since properness is stable under base change, it follows from 1.1.9 that $t_\Sigma(R)$ is proper, too. □

Combined with the Completion Theorem II.3.7.5 the above implies that every toric scheme can be openly embedded in a proper toric scheme "in a toric way".

(1.3.13) Corollary *Let Σ be a (simplicial) N-fan in V. Then, there exists a (simplicial) N-fan Σ' in V and an open immersion $u : X_\Sigma \to X_{\Sigma'}$ such that $X_{\Sigma'}(R)$ is proper over R for every ring R.*

Proof. Clear by II.3.7.5, 1.1.10 and 1.3.12. □

2. Cox schemes

Let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, let N be a $\mathbb{Z}$-structure on V, let $M := N^*$, and let Σ be an N-fan in V. If no confusion can arise, then we set $A := A_{\Sigma_1}$, $a := a_{\Sigma_1}$, and $c := c_{\Sigma_1}$, and if moreover Σ is full, then we consider $\mathrm{Pic}(\Sigma)$ *by means of b_Σ as a subgroup of A.*

2.1. Cox rings

We start by introducing some definitions concerning the lower standard sequence of a fan.

(2.1.1) It holds $\mathrm{Diff}(\mathbb{N}_0^{\Sigma_1}) = \mathbb{Z}^{\Sigma_1}$, and by means of the canonical morphism $\mathbb{N}_0^{\Sigma_1} \to \mathbb{Z}^{\Sigma_1}$ in Mon we consider $\mathbb{N}_0^{\Sigma_1}$ as a submonoid of $\mathbb{Z}^{\Sigma_1}$. Hence, we can and do consider every monoid of differences of $\mathbb{N}_0^{\Sigma_1}$ as a submonoid of $\mathbb{Z}^{\Sigma_1}$. If no confusion can arise, then we denote the canonical basis of the free monoid $\mathbb{N}_0^{\Sigma_1}$ by $(\delta_\rho)_{\rho\in\Sigma_1}$.

By means of restriction of $a : \mathbb{Z}^{\Sigma_1} \to A$ we consider submonoids of $\mathbb{Z}^{\Sigma_1}$, and in particular monoids of differences of $\mathbb{N}_0^{\Sigma_1}$, as monoids over A. We set $a^+_{\Sigma_1} := a\restriction_{\mathbb{N}_0^{\Sigma_1}} : \mathbb{N}_0^{\Sigma_1} \to A$, and if no confusion can arise we write $a^+ := a^+_{\Sigma_1}$. For $\rho \in \Sigma_1$ we set $\alpha_\rho := a(\delta_\rho) \in A$.

(2.1.2) For $\sigma \in \Sigma$ we set

$$\widehat{\delta}_\sigma := \sum_{\rho\in\Sigma_1\setminus\sigma_1} \delta_\rho \in \mathbb{N}_0^{\Sigma_1}$$

and $\widehat{\alpha}_\sigma := a(\widehat{\delta}_\sigma) \in A$. Then, for cones σ and τ in Σ we have $\widehat{\delta}_\tau \in \mathbb{N}_0^{\Sigma_1} + \widehat{\delta}_\sigma$ if and only if $\tau \preccurlyeq \sigma$, and then it holds

$$\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau = (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) - (\widehat{\delta}_\tau - \widehat{\delta}_\sigma)$$

and hence $\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma \subseteq \mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau$. Furthermore, the canonical injection $\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma \hookrightarrow \mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau$ and the canonical epimorphism

$$\varepsilon_{\widehat{\delta}_\tau - \widehat{\delta}_\sigma} : \mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma \to (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) - (\widehat{\delta}_\tau - \widehat{\delta}_\sigma)$$

are equal.

The monoideal of $\mathbb{N}_0^{\Sigma_1}$ generated by $\{\widehat{\delta}_\sigma \mid \sigma \in \Sigma\}$ is denoted by I_Σ and called *the irrelevant monoideal associated with* Σ. Obviously it is generated by $\{\widehat{\delta}_\sigma \mid \sigma \in \Sigma_{\max}\}$.

Now we define the Cox ring associated with a fan (or more precisely, with the set of its 1-dimensional cones)[3]. This makes use of the graded version of algebras of monoids as introduced in III.3.4.2. From the beginning on we carry out the whole construction relative to a subgroup B of the group A of degrees. Later on, B will be supposed to fulfil additional conditions (making

[3] This must not be confused with Cox rings as introduced by Hu and Keel in [15] (based also on [10]), although there are of course connections between the two notions.

it in some way "big enough"), and finally we specialise to $B = \mathrm{Pic}(\Sigma)$ (see 2.2.4 and 3.4.5).

(2.1.3) Let $B \subseteq A$ be a subgroup. Composing the extension functor

$$\bullet^{(A)} : \mathsf{Ann} \to \mathsf{GrAnn}^A$$

first with the functor

$$\bullet[a^+] : \mathsf{GrAnn}^A \to \mathsf{GrAnn}^A$$

under $\mathrm{Id}_{\mathsf{GrAnn}^A}$ that maps an A-graded ring R onto the A-graded algebra of $\mathbb{N}_0^{\Sigma_1}$ over R with respect to a^+ (see III.3.4.2) and then with the restriction functor $\bullet_{(B)} : \mathsf{GrAnn}^A \to \mathsf{GrAnn}^B$, yields a functor

$$S_{\Sigma_1,B}(\bullet) := \bullet^{(A_{\Sigma_1})}[a^+_{\Sigma_1}]_{(B)} : \mathsf{Ann} \to \mathsf{GrAnn}^B$$

under $\bullet^{(B)}$. Taking components of degree 0 yields a functor

$$S_{\Sigma_1,B}(\bullet)_0 : \mathsf{Ann} \to \mathsf{Ann}$$

under $\mathrm{Id}_{\mathsf{Ann}}$, and it is easy to see that

$$S_{\Sigma_1,B}(\bullet)_0 = \bullet[\mathrm{Ker}(a^+)]_{(B)} = \bullet[\mathrm{Im}(c) \cap \mathbb{N}_0^{\Sigma_1}]_{(B)}.$$

In case $B = A$ we write S_{Σ_1} instead of $S_{\Sigma_1,A}$. Clearly, we have $S_{\Sigma_1,B}(\bullet) = S_{\Sigma_1}(\bullet)_{(B)}$, and hence the functors $S_{\Sigma_1}(\bullet)_0$ and $S_{\Sigma_1,B}(\bullet)_0$ from Ann to Ann under $\mathrm{Id}_{\mathsf{Ann}}$ are equal.

(2.1.4) Let R be a ring. Then, $S_{\Sigma_1}(R)$ is called *the Cox ring associated with Σ_1 (and N) over R.* Its underlying R-algebra is the polynomial algebra over R in indeterminates $(Z_\rho)_{\rho\in\Sigma_1}$, and its A-graduation is given by $\deg(Z_\rho) = \alpha_\rho$ for every $\rho \in \Sigma_1$. If R' is a further ring and $h : R \to R'$ is a morphism in Ann, then the morphism $S_{\Sigma_1}(h) : S_{\Sigma_1}(R) \to S_{\Sigma_1}(R')$ in GrAnn^A is given by $Z_\rho \mapsto Z_\rho$ for every $\rho \in \Sigma_1$.

Now, let $B \subseteq R$ be a subgroup. Then, $S_{\Sigma,B}(R)$ is called *the B-restricted Cox ring associated with Σ (and N) over R.* Its underlying R-algebra is the sub-R-algebra of the polynomial algebra over R in indeterminates $(Z_\rho)_{\rho\in\Sigma_1}$ generated by the monomials whose degrees lie in B, that is, by $\mathbb{T}_{R,\Sigma,B} := \mathbb{T}_{R,\Sigma_1} \cap S_{\Sigma,B}(R)$.

(2.1.5) Example Let R be a ring, let $k \in \mathbb{N}$, and let Σ be as in II.4.2.2. Then, we have $A = \mathbb{Z}^2$, and $S_{\Sigma_1}(R)$ is the A-graded polynomial algebra $R[Z_1, Z_2, Z_3, Z_4]$ over R with $\deg(Z_1) = (0, 1)$, $\deg(Z_2) = (1, -k)$, $\deg(Z_3) = (0, 1)$ and $\deg(Z_4) = (1, 0)$.

The next result gives some information on Noetherianity of Cox rings.

(2.1.6) Proposition *Let $B \subseteq A$ be a subgroup, and let R be a ring.*

a) The R-algebra $S_{\Sigma_1,B}(R)$ is $\mathbb{T}_{R,\Sigma_1,B}$-Noetherian.

b) If the A-graded R-algebra $S_{\Sigma_1}(R)$ is Noetherian, then the B-graded R-algebra $S_{\Sigma,B}(R)$ is Noetherian.

Proof. a) holds by III.3.3.4 a), for it follows from I.1.2.8 and I.2.6.1 that $S_{\Sigma_1}(R)$ is $\mathbb{T}_{R,\Sigma_1}$-Noetherian. b) holds by III.3.3.5. □

The property of a Cox ring $S_{\Sigma_1,B}(R)$ to have a component of degree 0 coinciding with R depends only on the fan Σ, as we show next.

(2.1.7) Proposition *Let $B \subseteq A$ be a subgroup, and suppose that Σ is full. Then, the following statements are equivalent:*

(i) *Σ is skeletal complete;*

(ii) $S_{\Sigma_1,B}(\bullet)_0 = \mathrm{Id}_{\mathsf{Ann}}$*;*

(iii) *There is a ring $R \neq 0$ with $S_{\Sigma_1,B}(R)_0 = R$.*

Proof. By 2.1.3 we can suppose without loss of generality that $B = A$. Furthermore, we know from II.4.2.6 that Σ is skeletal complete if and only if $\mathrm{Im}(c) \cap \mathbb{N}_0^{\Sigma_1} = 0$, that is, if and only if $\mathrm{Ker}(a^+) = 0$. Hence, the equivalence of (i) and (ii) is clear by 2.1.3. So, suppose that there is a ring R with $R \neq 0$ such that $S_{\Sigma_1}(R)_0 = R$. Then, the R-module underlying $S_{\Sigma_1}(R)_0 = R[\mathrm{Ker}(a^+)]$ is free of rank 1, and from this we get $\mathrm{Im}(c) \cap \mathbb{N}_0^{\Sigma_1} = \mathrm{Ker}(a^+) = 0$ and thus again by II.4.2.6 that Σ is skeletal complete. □

(2.1.8) It is seen from its proof and II.4.2.7 that in 2.1.7 we cannot omit the hypothesis that Σ is full.

(2.1.9) Corollary *Let $B \subseteq A$ be a subgroup, let R be a ring, let F be a Noetherian B-graded $S_{\Sigma_1,B}(R)$-module, and let $\alpha \in B$. If Σ is skeletal complete, then the R-module F_α is Noetherian.*

Proof. Clear by III.3.3.6 and 2.1.7. □

We already know from the general results in Chapters I and III that Cox rings behave well under base change, but we state this formally for the sake of reference.

(2.1.10) Proposition *Let $B \subseteq A$ be a subgroup, let R be a ring, and let R' be an R-algebra. Then, there is a canonical isomorphism*

$$S_{\Sigma_1,B}(\bullet) \otimes_R R' \cong S_{\Sigma_1,B}(\bullet \otimes_R R')$$

of functors from $\mathsf{Alg}(R)$ to $\mathsf{GrAlg}^B(R')$ under $(\bullet \otimes_R R')^{(B)}$.

Proof. Clear by III.3.4.4. □

We end this section by introducing the irrelevant ideal of a Cox ring. This ideals will play a role similar to the irrelevant ideal in a positively graded ring defining a projective scheme.

(2.1.11) Let R be a ring. For $\sigma \in \Sigma$ we set

$$\widehat{Z}_\sigma := \exp_{R,\mathbb{N}_0^{\Sigma_1}}(\widehat{\delta}_\sigma) = \prod_{\rho \in \Sigma_1 \setminus \sigma_1} Z_\rho \in S_{\Sigma_1}(R),$$

and then it holds $\deg(\widehat{Z}_\sigma) = \widehat{\alpha}_\sigma$ (see 2.1.2). The A-graded ideal

$$I_\Sigma(R) := \mathrm{Exp}_{R,\mathbb{N}_0^{\Sigma_1}}(I_\Sigma) = \langle\{\widehat{Z}_\sigma \mid \sigma \in \Sigma\}\rangle_{S_{\Sigma_1}(R)} \subseteq S_{\Sigma_1}(R)$$

is called *the irrelevant ideal associated with Σ (and N) over R.* Clearly, it holds $I_\Sigma(R) = \langle\{\widehat{Z}_\sigma \mid \sigma \in \Sigma_{\max}\}\rangle_{S_{\Sigma_1}(R)}$, and $I_\Sigma(R)$ is finitely $\mathbb{T}_{R,\Sigma_1}$-generated.

Now, let $B \subseteq A$ be a subgroup. The B-graded ideal

$$I_{\Sigma,B}(R) := I_\Sigma(R) \cap S_{\Sigma,B}(R) = I_\Sigma(R)_{(B)} \subseteq S_{\Sigma,B}(R)$$

is called *the B-restricted irrelevant ideal associated with Σ (and N) over R;* it is finitely $\mathbb{T}_{R,\Sigma,B}$-generated by III.3.3.4 b) and 2.1.6 a).

2.2. Cox schemes

Our goal in this and the next section is to give another description of toric schemes. We start by defining a further projective system of monoids arising from a fan and also the so-called Cox scheme defined by this system.

(2.2.1) The family $((\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a))_{\sigma\in\Sigma}$ defines a projective system of submonoids of $\mathbb{Z}^{\Sigma_1}$ over the lower semilattice Σ, and this we denote by C_Σ.

Let $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$, and let $u \in \sigma_M^\vee$ with $\tau = \sigma \cap u^\perp$. Then, it clearly holds $c(u) \in (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a)$, and moreover the canonical injection

$$(\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a) \hookrightarrow (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau) \cap \mathrm{Ker}(a)$$

and the canonical morphism

$$\varepsilon_{c(u)} : (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a) \to ((\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a)) - c(u)$$

are equal. Indeed, there is a family $(u_\rho)_{\rho\in\Sigma_1}$ in $\mathbb{Z}$ with $c(u) = \sum_{\rho\in\Sigma_1} u_\rho\delta_\rho$, and it holds $u_\rho = 0$ for every $\rho \in \tau_1$ and $u_\rho > 0$ for every $\rho \in \sigma_1 \setminus \tau_1$. Hence, we have

$$-c(u) = \textstyle\sum_{\rho\in\Sigma_1\setminus\sigma_1} -u_\rho\delta_\rho + \sum_{\rho\in\sigma_1\setminus\tau_1} -u_\rho\delta_\rho \in \mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau$$

and

$$-\widehat{\delta}_\tau = \textstyle\sum_{\rho\in\sigma_1\setminus\tau_1}(u_\rho - 1)\delta_\rho + \sum_{\rho\in\Sigma_1\setminus\sigma_1} u_\rho\delta_\rho - \widehat{\delta}_\sigma - c(u) \in \mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma - c(u),$$

and therefore

$$(\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a) - c(u) \subseteq (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau) \cap \mathrm{Ker}(a) \subseteq$$

$$(\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma - c(u)) \cap \mathrm{Ker}(a) = (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a) - c(u).$$

This proves the claim.

The above and I.1.4.15 imply that the projective system C_Σ of submonoids of $\mathbb{Z}^\Sigma$ is openly immersive.

(2.2.2) Keeping in mind 2.2.1 we apply the construction of I.1.4.10 to the projective system C_Σ of submonoids of $\mathbb{Z}^\Sigma$ to get in $\mathsf{Hom}(\mathsf{Ann}^\circ, \mathsf{Sch})$ a morphism

$$t_{C_\Sigma} : X_{C_\Sigma} \to \mathrm{Spec},$$

for every $\sigma \in \Sigma$ a morphism $t_{C_\Sigma,\sigma} : X_{C_\Sigma,\sigma} \to \mathrm{Spec}$ and an open immersion $\iota_{C_\Sigma,\sigma} : X_{C_\Sigma,\sigma} \to X_{C_\Sigma}$, and for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ an open immersion $\iota_{C_\Sigma,\tau,\sigma} : X_{C_\Sigma,\tau} \to X_{C_\Sigma,\sigma}$ such that if R is a ring, then $X_{C_\Sigma}(R)$ is obtained by glueing the family $(X_{C_\Sigma,\sigma}(R))_{\sigma\in\Sigma}$ of R-schemes along $(X_{C_\Sigma,\sigma\cap\tau})_{(\sigma,\tau)\in\Sigma^2}$. If no confusion can arise, then we set $Y_\Sigma := X_{C_\Sigma}$ and $u_\Sigma := t_{C_\Sigma}$, $Y_\sigma := X_{C_\Sigma,\sigma}$, $u_\sigma := t_{C_\Sigma,\sigma}$ and $\kappa_\sigma := \iota_{C_\Sigma,\sigma}$ for $\sigma \in \Sigma$, and $\kappa_{\tau,\sigma} := \iota_{C_\Sigma,\tau,\sigma}$ for $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$.

If R is a ring, then the R-scheme $u_\Sigma(R) : Y_\Sigma(R) \to \mathrm{Spec}(R)$, and by abuse of language also its underlying scheme $Y_\Sigma(R)$, is called *the Cox scheme over R associated with Σ (and N).*

Next, we introduce an additional condition on the subgroup $B \subseteq A$.

(2.2.3) For $\sigma \in \Sigma$, we denote by $\widehat{A}_\Sigma$ or, if no confusion can arise by $\widehat{A}$, the subgroup of A generated by $\{\widehat{\alpha}_\sigma \mid \sigma \in \Sigma\}$.

A subgroup $B \subseteq A$ is called Σ-*big (in A)* or, if no confusion can arise, *big (in A)* if it has finite index in $\widehat{A} + B$, that is, if for every $\sigma \in \Sigma$ there is an $m_\sigma \in \mathbb{N}$ such that $m\widehat{\alpha}_\sigma \in B$.

(2.2.4) Example Obviously, A_{Σ_1} is Σ-big in A_{Σ_1}. If Σ is full and simplicial, then II.4.2.9 shows that $\mathrm{Pic}(\Sigma)$ is Σ-big in A_{Σ_1}.

The following description of Cox schemes generalises the description of projective schemes in [ÉGA, II.2.3–4].

(2.2.5) Let $B \subseteq A$ be a big subgroup, let $\sigma \in \Sigma$, and let $m \in \mathbb{N}$ be such that $m\widehat{\alpha}_\sigma \in B$. The functor

$$S_{\Sigma_1,B}(\bullet)_{\widehat{Z}_\sigma^m} : \mathsf{Ann} \to \mathsf{GrAnn}^A$$

under $\bullet^{(B)}$ and the canonical morphism

$$\eta_{\widehat{Z}_\sigma^m} : S_{\Sigma_1,B}(\bullet) \to S_{\Sigma_1,B}(\bullet)_{\widehat{Z}_\sigma^m}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{GrAnn}^B)^{/\bullet^{(B)}}$ are independent of the choice of m. We denote them by $S_{\Sigma_1,B}(\bullet)_\sigma$ and η_σ, respectively. Composition with the functor of taking components of degree 0 from GrAnn^B to Ann yields a functor

$$S_{\Sigma_1,B}(\bullet)_{(\sigma)} := S_{\Sigma_1,B}(\bullet)_{(\widehat{Z}_\sigma^m)} : \mathsf{Ann} \to \mathsf{Ann}$$

under $\mathrm{Id}_{\mathsf{Ann}}$ and a morphism

$$\eta_{(\sigma)} : S_{\Sigma_1,B}(\bullet)_0 \to S_{\Sigma_1,B}(\bullet)_{(\sigma)}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Ann})^{/\mathrm{Id}_{\mathsf{Ann}}}$.

It follows from III.2.5.11 that $S_{\Sigma_1,B}(\bullet)_\sigma = (S_{\Sigma_1}(\bullet)_\sigma)_{(B)}$, and moreover we have $S_{\Sigma_1,B}(\bullet)_{(\sigma)} = S_{\Sigma_1}(\bullet)_{(\sigma)}$.

(2.2.6) Let $B \subseteq A$ be a big subgroup, let $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$, and let $m \in \mathbb{N}$ be such that $m\widehat{\alpha}_\sigma \in B$ and $m\widehat{\alpha}_\tau \in B$. Then, $\widehat{Z}_\sigma^m$ divides $\widehat{Z}_\tau^m$ in $S_{\Sigma_1,B}(R)$ for every ring R, and hence we have a morphism

$$\eta_\tau^\sigma : S_{\Sigma_1,B}(\bullet)_\sigma \to S_{\Sigma_1,B}(\bullet)_\tau$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{GrAnn}^B)^{/\bullet^{(B)}}$, mapping a ring R onto the canonical morphism

$$\eta_{\widehat{Z}_\tau}^{\widehat{Z}_\sigma}(S_{\Sigma_1}(R)) : S_{\Sigma_1}(R)_{\widehat{Z}_\sigma} \to S_{\Sigma_1}(R)_{\widehat{Z}_\tau}$$

in $\mathsf{GrAlg}^A(R)$ and being independent of the choice of m. Composition with the functor $\bullet_0 : \mathsf{GrAnn}^B \to \mathsf{Ann}$ of taking components of degree 0 yields a morphism

$$\eta_{(\tau)}^{(\sigma)} : S_{\Sigma_1,B}(\bullet)_{(\sigma)} \to S_{\Sigma_1,B}(\bullet)_{(\tau)}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Ann})^{/\mathrm{Id}_{\mathsf{Ann}}}$.

(2.2.7) Let $B \subseteq A$ be a big subgroup. It follows from I.1.3.14 and 2.2.5 that for every $\sigma \in \Sigma$ there is a canonical isomorphism

$$\varepsilon_\sigma : \bullet[\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma]_{(B)} \xrightarrow{\cong} S_{\Sigma_1,B}(\bullet)_\sigma$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{GrAnn}^B)^{/\bullet^{(B)}}$ such that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
\bullet[\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma]_{(B)} \arrow[r, "\varepsilon_\sigma", "\cong"'] \arrow[d] & S_{\Sigma_1,B}(\bullet)_\sigma \arrow[d, "\eta_\tau^\sigma"] \\
\bullet[\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau]_{(B)} \arrow[r, "\varepsilon_\tau", "\cong"'] & S_{\Sigma_1,B}(\bullet)_\tau
\end{tikzcd}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{GrAnn}^B)^{/\bullet^{(B)}}$, where the unmarked morphism is induced by the canonical injection $\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma \hookrightarrow \mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau$, commutes. Taking components of degree 0 and keeping in mind 2.1.3 we get for every $\sigma \in \Sigma$ a canonical isomorphism

$$\varepsilon_{(\sigma)} : \bullet[(\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a)] \xrightarrow{\cong} S_{\Sigma_1,B}(\bullet)_{(\sigma)}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Ann})^{/\mathrm{Id}_{\mathsf{Ann}}}$ such that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
\bullet[(\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a)] \arrow[r, "\varepsilon_{(\sigma)}", "\cong"'] \arrow[d] & S_{\Sigma_1,B}(\bullet)_{(\sigma)} \arrow[d, "\eta_{(\tau)}^{(\sigma)}"] \\
\bullet[(\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau) \cap \mathrm{Ker}(a)] \arrow[r, "\varepsilon_{(\tau)}", "\cong"'] & S_{\Sigma_1,B}(\bullet)_{(\tau)}
\end{tikzcd}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Ann})^{/\mathrm{Id}_{\mathsf{Ann}}}$, where the unmarked morphism is induced by the canonical injection $(\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a) \hookrightarrow (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau) \cap \mathrm{Ker}(a)$, commutes.

(2.2.8) Let $B \subseteq A$ be a big subgroup. If $\sigma \in \Sigma$, then by 2.2.7 we can and do identify $u_\sigma : Y_\sigma \to \mathrm{Spec}$ with the composition of Spec with the canonical injection $\mathrm{Id}_{\mathsf{Ann}}(\bullet) \to S_{\Sigma_1,B}(\bullet)_{(\sigma)}$. It follows that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the open immersion $\kappa_{\tau,\sigma} : Y_\tau \to Y_\sigma$ is identified with $\mathrm{Spec} \circ \eta_{(\tau)}^{(\sigma)}$.

So, for a ring R we obtain the R-scheme $Y_\Sigma(R)$ also by glueing the family $(\mathrm{Spec}(S_{\Sigma_1,B}(R)_{(\sigma)}))_{\sigma\in\Sigma}$ of R-schemes along $(\mathrm{Spec}(S_{\Sigma_1,B}(R)_{(\sigma\cap\tau)}))_{(\sigma,\tau)\in\Sigma^2}$, and as is also seen by 2.2.5 this construction is independent of the choice of the big subgroup $B \subseteq A$.

2.3. Cox schemes and toric schemes

Now we construct a natural morphism from the Cox scheme $Y_\Sigma(R)$ to the toric scheme $X_\Sigma(R)$, and we show that this is an isomorphism if and only if the fan Σ is full. Together with the base change results given above this allows us to apply results on Cox schemes to toric schemes and vice versa.

(2.3.1) The morphism $c : M \to \mathbb{Z}^{\Sigma_1}$ induces by restriction and coastriction for every $\sigma \in \Sigma$ a surjective morphism

$$c_\sigma : \sigma_M^\vee \to (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a)$$

in Mon such that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
\sigma_M^\vee \arrow[r, "c_\sigma"] \arrow[d, hook] & (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap \mathrm{Ker}(a) \arrow[d, hook] \\
\tau_M^\vee \arrow[r, "c_\tau"] & (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\tau) \cap \mathrm{Ker}(a)
\end{tikzcd}$$

in Mon commutes.

So, by 2.2.7 we have for every $\sigma \in \Sigma$ a canonical morphism

$$c'_\sigma : \bullet[\sigma_M^\vee] \to S_{\Sigma_1}(\bullet)_{(\sigma)}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Ann})^{/\mathrm{Id}_{\mathsf{Ann}}}$ such that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
\bullet[\sigma_M^\vee] \arrow[r, "c'_\sigma"] \arrow[d] & S_{\Sigma_1}(\bullet)_{(\sigma)} \arrow[d, "\eta^{(\sigma)}_{(\tau)}"] \\
\bullet[\tau_M^\vee] \arrow[r, "c'_\sigma"] & S_{\Sigma_1}(\bullet)_{(\tau)}
\end{tikzcd}$$

in $\mathsf{Hom}(\mathsf{Ann}, \mathsf{Ann})^{/\mathrm{Id}_{\mathsf{Ann}}}$, where the unmarked morphism is induced by the canonical injection $\sigma_M^\vee \hookrightarrow \tau_M^\vee$, commutes.

Taking spectra yields for every $\sigma \in \Sigma$ a canonical morphism $\gamma_\sigma := \mathrm{Spec}(c'_\sigma) : Y_\sigma \to X_\sigma$ in $\mathsf{Hom}(\mathsf{Ann}^\circ, \mathsf{Sch})_{/\,\mathrm{Spec}}$ such that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
Y_\sigma \arrow[r, "\gamma_\sigma"] & X_\sigma \\
Y_\tau \arrow[r, "\gamma_\tau"] \arrow[u, "\mathrm{Spec}(\eta^{(\sigma)}_{(\tau)})"] & X_\tau \arrow[u, "\iota_{\tau,\sigma}"']
\end{tikzcd}$$

in $\mathsf{Hom}(\mathsf{Ann}^\circ, \mathsf{Sch})_{/\,\mathrm{Spec}}$ commutes.

Moreover, if $\sigma \in \Sigma$, then c_σ and hence also c'_σ is surjective, and therefore γ_σ is a closed immersion.

(2.3.2) Proposition *There exists a unique morphism*

$$\gamma_\Sigma : Y_\Sigma \to X_\Sigma$$

in $\mathsf{Hom}(\mathsf{Ann}^\circ, \mathsf{Sch})_{/\,\mathrm{Spec}}$ *such that for every* $\sigma \in \Sigma$ *it holds* $\iota_\sigma \circ \gamma_\sigma = \gamma_\Sigma \circ \kappa_\sigma$.

Proof. This follows from the construction of Y_Σ and 2.3.1. □

(2.3.3) Proposition *The fan* Σ *is full if and only if* $\gamma_\Sigma : Y_\Sigma \to X_\Sigma$ *is an isomorphism.*

Proof. It follows from II.4.2.3 that

$$\mathrm{Ker}(c_\sigma) = M \cap \langle\Sigma\rangle^\perp \cap \sigma_M^\vee = M \cap \langle\Sigma\rangle^\perp$$

for every $\sigma \in \Sigma$. Therefore, Σ is full if and only if c_σ is an isomorphism for every $\sigma \in \Sigma$, hence if and only if c'_σ is an isomorphism for every $\sigma \in \Sigma$. This is equivalent to γ_σ being an isomorphism for every $\sigma \in \Sigma$, and this clearly holds if and only if γ_Σ is an isomorphism. □

(2.3.4) If Σ is full, then by 2.3.3 we can and do identify Y_Σ and X_Σ by means of the canonical isomorphism γ_Σ. So, the results on Cox schemes in this chapter apply to toric schemes associated with full fans.

On the other hand, suppose that Σ is not full. We denote by Σ' the set Σ considered as a full fan in $\langle\Sigma\rangle$, and we set $N' := N \cap \langle\Sigma\rangle$. If R is a ring, then by 1.1.11 and the above we have a (non-canonical) isomorphism $X_\Sigma(R) \cong X_{\Sigma'}(R[N/N']) = Y_{\Sigma'}(R[N/N'])$ of schemes, and hence the results on Cox schemes apply to toric schemes associated with nonfull fans after a suitable base change.

3. Sheaves of modules on toric schemes

Let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, let N be a $\mathbb{Z}$-structure on V, let $M := N^*$, and let Σ be an N-fan in V. If no confusion can arise, then we set $A := A_{\Sigma_1}$, $P := \mathrm{Pic}(\Sigma)$, $c := c_{\Sigma_1}$, $d := d_{\Sigma}$, and $e := e_{\Sigma}$, and if moreover Σ is full, then we consider P by means of b_{Σ} as a subgroup of A. Furthermore, let R be a ring. If no confusion can arise, then we set $S := S_{\Sigma_1}(R)$, $Y_{\Sigma} := Y_{\Sigma}(R)$, $Y_{\sigma} := Y_{\sigma}(R)$ for $\sigma \in \Sigma$, and $\kappa_{\tau,\sigma} := \kappa_{\tau,\sigma}(R)$ for $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$.

3.1. Sheaves associated with graded modules

Let $B \subseteq A$ be a big subgroup. If no confusion can arise, then we set $S_{B,\sigma} := (S_{(B)})_{\sigma}$.

A fundamental fact of algebraic geometry is the equivalence between the category of modules over a ring T and the category of quasicoherent sheaves of modules on the scheme $\mathrm{Spec}(T)$ ([ÉGA, I.1.4.2]). Moreover, these equivalence gives rise to a functor from the category of $\mathbb{Z}$-graded modules over a positively graded ring T to the category of quasicoherent sheaves of modules on the scheme $\mathrm{Proj}(T)$. If T is generated as a T_0-algebra by finitely many elements of degree 1, then this functor is essentially surjective, but not necessarily injective. However, it induces a bijection from the set of graded ideals of T that are saturated with respect to the irrelevant ideal T_+ of T to the set of quasicoherent ideals of $\mathscr{O}_{\mathrm{Proj}(T)}$ ([ÉGA, II.2.5.2; II.2.7.7; II.2.7.3]).

We are going to elaborate analogues of the above for toric schemes and hence generalise results of Cox ([10]) and Mustaţă ([18]) for toric varieties. The role of T is played by the Cox ring, and so we start by defining the sheaf of modules associated with a graded module over the Cox ring.

(3.1.1) For every $\sigma \in \Sigma$ we have the functors

$$\bullet_{\sigma} := \bullet \otimes_{S_{(B)}} S_{B,\sigma} : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{GrMod}^B(S_{B,\sigma})$$

and

$$\bullet_{(\sigma)} := (\bullet_{\sigma})_0 : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{Mod}(S_{(\sigma)}).$$

For $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ we have functors

$$\bullet \otimes_{S_{B,\sigma}} S_{B,\tau} = (\eta^{\sigma}_{\tau})^*(\bullet) : \mathsf{GrMod}^B(S_{B,\sigma}) \to \mathsf{GrMod}^B(S_{B,\tau})$$

and

$$\bullet \otimes_{S_{(\sigma)}} S_{(\tau)} = (\eta^{(\sigma)}_{(\tau)})^*(\bullet) : \mathsf{Mod}(S_{(\sigma)}) \to \mathsf{Mod}(S_{(\tau)})$$

such that the diagram

$$\begin{tikzcd}
\mathsf{GrMod}^B(S_{(B)}) \arrow[r, "\bullet_{\sigma}"] \arrow[dr, "\bullet_{\tau}"'] & \mathsf{GrMod}^B(S_{B,\sigma}) \arrow[r, "\bullet_0"] \arrow[d, "(\eta^{\sigma}_{\tau})^*"] & \mathsf{Mod}(S_{(\sigma)}) \arrow[d, "(\eta^{(\sigma)}_{(\tau)})^*"] \\
 & \mathsf{GrMod}^B(S_{B,\tau}) \arrow[r, "\bullet_0"] & \mathsf{Mod}(S_{(\tau)})
\end{tikzcd}$$

of categories commutes.

(3.1.2) Let $\sigma \in \Sigma$. Then, by [ÉGA, I.1.3.4; I.1.4.1] we have a functor

$$\mathsf{Mod}(S_{(\sigma)}) \to \mathsf{Mod}(\mathscr{O}_{Y_\sigma})$$

that maps an $S_{(\sigma)}$-module F onto the $\mathscr{O}_{Y_\sigma}$-module $\widetilde{F}$ associated with F which is quasicoherent. We denote this functor and also its coastriction to $\mathsf{QCMod}(\mathscr{O}_{Y_\sigma})$ by $\mathscr{S}_{\sigma,R}$ or, if no confusion can arise, by $\mathscr{S}_\sigma$. Conversely, taking global sections yields a functor

$$\mathsf{Mod}(\mathscr{O}_{Y_\sigma}) \to \mathsf{Mod}(S_{(\sigma)}).$$

We denote this functor and also its restriction to $\mathsf{QCMod}(\mathscr{O}_{Y_\sigma})$ by $\Gamma(Y_\sigma, \bullet)$. From [ÉGA, I.1.4.2] we know that the above functors

$$\mathscr{S}_\sigma : \mathsf{Mod}(S_{(\sigma)}) \to \mathsf{QCMod}(\mathscr{O}_{Y_\sigma})$$

and

$$\Gamma(Y_\sigma, \bullet) : \mathsf{QCMod}(\mathscr{O}_{Y_\sigma}) \to \mathsf{Mod}(S_{(\sigma)})$$

are mutually quasiinverse and hence define an equivalence of categories between $\mathsf{Mod}(S_{(\sigma)})$ and $\mathsf{QCMod}(\mathscr{O}_{Y_\sigma})$.

(3.1.3) Proposition *There exists a unique functor*

$$\mathscr{S}_{\Sigma,B,R} : \mathsf{GrMod}^B(S) \to \mathsf{Mod}(\mathscr{O}_{Y_\Sigma})$$

such that for every $\sigma \in \Sigma$ the diagram

$$\begin{tikzcd}
\mathsf{GrMod}^B(S_{(B)}) \arrow[r, "\mathscr{S}_{\Sigma,B,R}"] \arrow[d, "\bullet_{(\sigma)}"'] & \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}) \arrow[d, "\upharpoonright_{Y_\sigma}"] \\
\mathsf{Mod}(S_{(\sigma)}) \arrow[r, "\mathscr{S}_{\sigma,R}"] & \mathsf{Mod}(\mathscr{O}_{Y_\sigma})
\end{tikzcd}$$

of categories commutes, and $\mathscr{S}_{\Sigma,B,R}$ is exact and commutes with inductive limits.

Proof. Let $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$. Then, by [ÉGA, I.1.7.7] the diagram

$$\begin{tikzcd}
\mathsf{Mod}(S_{(\sigma)}) \arrow[r, "\mathscr{S}_\sigma"] \arrow[d, "(\eta^{(\sigma)}_{(\tau)})^*"'] & \mathsf{Mod}(\mathscr{O}_{Y_\sigma}) \arrow[d, "\kappa^*_{\tau,\sigma}"] \\
\mathsf{Mod}(S_{(\tau)}) \arrow[r, "\mathscr{S}_\tau"] & \mathsf{Mod}(\mathscr{O}_{Y_\tau})
\end{tikzcd}$$

of categories commutes. Hence, on use of [ÉGA, 0.3.3.1–2] the existence of $\mathscr{S}_{\Sigma,B,R}$ follows by glueing. The remaining claims follow immediately from the construction and [ÉGA, I.1.3.9]. □

(3.1.4) If no confusion can arise, then we write $\mathscr{S}_{\Sigma,B}$ instead of $\mathscr{S}_{\Sigma,B,R}$, and $\mathscr{S}_\Sigma$ instead of $\mathscr{S}_{\Sigma,A}$. If F is a B-graded $S_{(B)}$-module, then $\mathscr{S}_{\Sigma,B}(F)$ is called *the $\mathscr{O}_{Y_\Sigma}$-module associated with F.* Clearly, $\mathscr{S}_{\Sigma,B}(F)$ is quasicoherent. Hence, $\mathscr{S}_{\Sigma,B}$ induces by coastriction a functor from $\mathsf{GrMod}^B(S_{(B)})$ to $\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma})$ that is exact and commutes with inductive limits, and we denote this also by $\mathscr{S}_{\Sigma,B,R}$ or, if no confusion can arise, by $\mathscr{S}_{\Sigma,B}$, and in case $B = A$ by $\mathscr{S}_{\Sigma}$.

The next result shows that the above construction commutes with restriction of degrees to the big subgroup $B \subseteq A$.

(3.1.5) Corollary *The diagram*

$$\begin{tikzcd} \mathsf{GrMod}^A(S) \arrow[r, "\mathscr{S}_\Sigma"] \arrow[d, "\bullet_{(B)}"'] & \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}) \\ \mathsf{GrMod}^B(S_{(B)}) \arrow[ur, "\mathscr{S}_{\Sigma,B}"'] & \end{tikzcd}$$

of categories commutes.

Proof. This is immediately clear by III.1.4.3 and 3.1.3. □

By construction, the functors $\mathscr{S}_{\Sigma,B}$ share some properties with their "affine versions" $\mathscr{S}_\sigma$ that we will state next.

(3.1.6) If $\sigma \in \Sigma$, then $\mathscr{S}_\sigma(S_{(\sigma)}) = \mathscr{O}_{Y_\sigma}$ is a sheaf of rings on Y_σ. This defines on $\mathscr{S}_{\Sigma,B}(S_{(B)})$ a structure of sheaf of rings on Y_Σ, and we always furnish $\mathscr{S}_{\Sigma,B}(S_{(B)})$ with this structure. Then, it clearly holds $\mathscr{S}_{\Sigma,B}(S_{(B)}) = \mathscr{O}_{Y_\Sigma}$.

(3.1.7) Let F be a B-graded $S_{(B)}$-module. If $G \subseteq F$ is a graded sub-$S_{(B)}$-module and $j : G \hookrightarrow F$ denotes the canonical injection, then

$$\mathscr{S}_{\Sigma,B}(j) : \mathscr{S}_{\Sigma,B}(G) \to \mathscr{S}_{\Sigma,B}(F)$$

is a monomorphism in $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma})$ since $\mathscr{S}_{\Sigma,B}$ is exact by 3.1.3. In this case we identify $\mathscr{S}_{\Sigma,B}(j)$ with its image and hence consider $\mathscr{S}_{\Sigma,B}(G)$ as a quasicoherent sub-$\mathscr{O}_{Y_\Sigma}$-module of $\mathscr{S}_{\Sigma,B}(F)$.

Furthermore, exactness of $\mathscr{S}_{\Sigma,B}$ shows that if $H \subseteq F$ is a further graded sub-$S_{(B)}$-module, then it holds

$$\mathscr{S}_{\Sigma,B}(G + H) = \mathscr{S}_{\Sigma,B}(G) + \mathscr{S}_{\Sigma,B}(H)$$

and

$$\mathscr{S}_{\Sigma,B}(G \cap H) = \mathscr{S}_{\Sigma,B}(G) \cap \mathscr{S}_{\Sigma,B}(H).$$

(3.1.8) Proposition *Let $h : R \to R'$ be an R-algebra. Then, there exists a canonical isomorphism*

$$Y_\Sigma(h)^*(\mathscr{S}_{\Sigma,B,R}(\bullet)) \cong \mathscr{S}_{\Sigma,B,R'}(\bullet \otimes_R R')$$

of functors from $\mathsf{GrMod}^B(S_{\Sigma_1,B}(R))$ to $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma(R')})$.

Proof. We consider the diagram

$$\begin{tikzcd}
& \mathsf{GrMod}^B(S(R)_{(B)}) \arrow[rr, "\mathscr{S}_{\Sigma,B,R}"] \arrow[dl, "\bullet\otimes_R R'"'] \arrow[dd, "\bullet_{(\sigma)}"] & & \mathsf{Mod}(\mathscr{O}_{Y_\Sigma(R)}) \arrow[dl, "\bullet\otimes_R R'"] \arrow[dd, "\upharpoonright_{Y_\sigma(R)}"] \\
\mathsf{GrMod}^B(S(R')_{(B)}) \arrow[rr, "\mathscr{S}_{\Sigma,B,R'}"] \arrow[dd, "\bullet_{(\sigma)}"'] & & \mathsf{Mod}(\mathscr{O}_{Y_\Sigma(R')}) \arrow[dd, "\upharpoonright_{Y_\sigma(R')}"] & \\
& \mathsf{Mod}(S(R)_{(\sigma)}) \arrow[rr, "\mathscr{S}_{\sigma,R}"] \arrow[dl, "\bullet\otimes_R R'"'] & & \mathsf{Mod}(\mathscr{O}_{Y_\sigma(R)}) \arrow[dl, "\bullet\otimes_R R'"] \\
\mathsf{Mod}(S(R')_{(\sigma)}) \arrow[rr, "\mathscr{S}_{\sigma,R'}"'] & & \mathsf{Mod}(\mathscr{O}_{Y_\sigma(R')}) &
\end{tikzcd}$$

of categories. This diagrams front and back commute by 3.1.3, its right side commutes by construction of Y_Σ, and its bottom commutes by [ÉGA, I.1.7.7] and 2.1.10. Furthermore, it is readily checked that its left side commutes, and hence its top commutes, too. Thus, the claim is proven. □

(3.1.9) Proposition *Let F be a Noetherian B-graded $S_{(B)}$-module.*

a) The $\mathscr{O}_{Y_\Sigma}$-module $\mathscr{S}_{\Sigma,B}(F)$ is of finite type.

b) If R is Noetherian, then $\mathscr{S}_{\Sigma,B}(F)$ is coherent.

Proof. a) For every $\sigma \in \Sigma$ the B-graded $S_{B,\sigma}$-module F_σ is Noetherian, and hence the $S_{(\sigma)}$-module $F_{(\sigma)}$ is Noetherian and in particular finitely generated by III.3.3.6. Therefore, $\mathscr{S}_{\Sigma,B}(F)$ is of finite type by [ÉGA, I.1.4.3] and 3.1.3.

b) is clear by a), I.2.6.4 b) and [ÉGA, I.2.7.1]. □

Similar to shifting of graded modules we introduce functors $\bullet(\alpha)$ that "twist" sheaves on Cox schemes. A word of warning may be appropriate here, for the connections between shifting and twisting on Cox schemes is more complicated than in the case homogeneous spectra defined by positively graded rings ([ÉGA, II.2.5.13]). We will see an example of bad behaviour later in 4.1.6.

(3.1.10) Let $\alpha \in A$. It follows from 3.1.5 that if $\alpha \in B$, then we have $\mathscr{S}_{\Sigma,B}(S_{(B)}(\alpha)) = \mathscr{S}_\Sigma(S(\alpha))$.

If $\mathscr{F}$ is an $\mathscr{O}_{Y_\Sigma}$-module, then we set

$$\mathscr{F}(\alpha) := \mathscr{S}_\Sigma(S(\alpha)) \otimes_{\mathscr{O}_{Y_\Sigma}} \mathscr{F},$$

and thus we get a functor

$$\bullet(\alpha) : \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}) \to \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}).$$

We know from [ÉGA, 0.4.1.5] that this functor is right exact and that $\mathscr{O}_{Y_\Sigma}(\alpha) = \mathscr{S}_\Sigma(S(\alpha))$.

(3.1.11) Proposition *Suppose that Σ is full, let $\sigma \in \Sigma$, and let $\alpha \in P$. Then, there exists a unique virtual polytope $p = (m_\tau + \tau^\vee)_{\tau \in \Sigma}$ over Σ such that $m_\tau = 0$ for every $\tau \in \mathrm{face}(\sigma)$ and that $e(p) = \alpha$, and the $S_{(\sigma)}$-module $S(\alpha)_{(\sigma)}$ is free of rank 1 with basis $\prod_{\rho \in \Sigma_1} Z_\rho^{\rho_N(m_\rho)}$.*

Proof. Existence of a unique p with the required properties holds by II.4.1.9. Furthermore, we have

$$d(p) = \sum_{\rho \in \Sigma_1} \rho_N(m_\rho)\delta_\rho \in (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap c^{-1}(\alpha),$$

and hence there exists for every $q \in (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap c^{-1}(\alpha)$ an $m \in M$ with $q = c(m) + d(p)$, that is, with

$$\prod_{\rho \in \Sigma_1} Z_\rho^{q_\rho} = (\prod_{\rho \in \Sigma_1} Z_\rho^{\rho_N(m)})(\prod_{\rho \in \Sigma_1} Z_\rho^{\rho_N(m_\rho)})$$

in S_σ. Since every element of $S(\alpha)_{(\sigma)} = (S_\sigma)_\alpha$ is an $S_{(\sigma)}$-linear combination of elements of the form $\prod_{\rho \in \Sigma_1} Z_\rho^{q_\rho}$ with $q \in (\mathbb{N}_0^{\Sigma_1} - \widehat{\delta}_\sigma) \cap c^{-1}(\alpha)$, this implies that $\prod_{\rho \in \Sigma_1} Z_\rho^{\rho_N(m_\rho)}$ generates the $S_{(\sigma)}$-module $S(\alpha)_\sigma$. As this element is obviously free over $S_{(\sigma)}$ the claim is proven. $\square$

(3.1.12) Corollary *Suppose that Σ is full, and let $\alpha \in P$. Then, the $\mathscr{O}_{Y_\Sigma}$-module $\mathscr{O}_{Y_\Sigma}(\alpha)$ is invertible.*

Proof. Clear from 3.1.11, since $\mathscr{O}_{Y_\Sigma}(\alpha) \restriction_{Y_\sigma} = \mathscr{S}_\sigma(S(\alpha)_{(\sigma)})$ for every $\sigma \in \Sigma$. $\square$

(3.1.13) Corollary *Suppose that Σ is full, let $\sigma \in \Sigma$, and let $\alpha, \beta \in P$. Then, the morphism*

$$S(\alpha)_{(\sigma)} \otimes_{S_{(\sigma)}} S(\beta)_{(\sigma)} \to S(\alpha + \beta)_{(\sigma)}$$

in $\mathsf{Mod}(S_{(\sigma)})$ induced by multiplication of S_σ is an isomorphism.

Proof. The morphism in question is induced by restriction and coastriction of the canonical isomorphism $S_\sigma \otimes_{S_\sigma} S_\sigma \to S_\sigma$ and hence injective. By 3.1.11 there are virtual polytopes $p = (m_\tau + \tau^\vee)_{\tau \in \Sigma}$ and $q = (l_\tau + \tau^\vee)_{\tau \in \Sigma}$ over Σ such that $e_\Sigma(p) = \alpha$ and $e_\Sigma(q) = \beta$, and that $m_\tau = 0$ and $l_\tau = 0$ for every $\tau \in \mathrm{face}(\sigma)$. This implies $p + q = (m_\tau + l_\tau + \tau^\vee)_{\tau \in \Sigma}$ and $e_\Sigma(p + q) = \alpha + \beta$, and then the claim follows easily on use of 3.1.11. $\square$

Now we can show that some rings of fractions of the P-restriction of S are strongly graded (see III.3.1).

(3.1.14) Corollary *Suppose that Σ is full and simplicial, and let $\sigma \in \Sigma$. Then, the P-graded ring $S_{P,\sigma}$ is strongly graded.*

Proof. Since Σ is full and simplicial, the subgroup $P \subseteq A$ is big by 2.2.4. For every $\alpha \in P$ the morphism

$$(S_\sigma)_\alpha \otimes_{S_{(\sigma)}} (S_\sigma)_{-\alpha} \to S_{(\sigma)}$$

in $\mathsf{Mod}(S_{(\sigma)})$ induced by multiplication of S_σ is an isomorphism by 3.1.13, and thus the claim follows from III.3.1.3. □

(3.1.15) Corollary *Suppose that Σ is full and simplicial, let F be a P-graded $S_{(P)}$-module, let $\sigma \in \Sigma$, and let $\alpha \in P$. Then, there is an isomorphism $(F_\sigma)_\alpha \cong F_{(\sigma)}$ in $\mathsf{Mod}(S_{(\sigma)})$.*

Proof. On use of III.2.5.11 we get $(S_{P,\sigma})_\beta = ((S_\sigma)_{(P)})_\beta = (S_\sigma)_\beta$ for every $\beta \in P$. If we apply this with $\beta = \alpha$ and $\beta = 0$ and keep in mind 3.1.11, then we get an isomorphism $(S_{P,\sigma})_\alpha \cong S_{(\sigma)}$ in $\mathsf{Mod}(S_{(\sigma)})$. Moreover, it follows from 3.1.14 and III.3.1.3 that $F_\sigma \cong S_{P,\sigma} \otimes_{S_{(\sigma)}} F_{(\sigma)}$ in $\mathsf{Mod}(S_{(\sigma)})$. Thus, taking components of degree α yields

$$(F_\sigma)_\alpha \cong (S_{P,\sigma} \otimes_{S_{(\sigma)}} F_{(\sigma)})_\alpha = (S_\sigma)_\alpha \otimes_{S_{(\sigma)}} F_{(\sigma)} \cong S_{(\sigma)} \otimes_{S_{(\sigma)}} F_{(\sigma)} \cong F_{(\sigma)}$$

in $\mathsf{Mod}(S_{(\sigma)})$ and hence the claim. □

From the above results we derive now quickly a flatness criterion for sheaves of the form $\mathscr{S}_{\Sigma,B}(F)$ in terms of the graded module F. In order to characterise flatness of such sheaves in 3.1.17 we have to suppose that Σ is simplicial and that $B = P$.

(3.1.16) Proposition *Let F be a B-graded $S_{(B)}$-module.*

a) $\mathscr{S}_{\Sigma,B}(F)$ is flat over R if and only if $F_{(\sigma)}$ is flat over R for every $\sigma \in \Sigma$.

b) If F_σ is flat over R for every $\sigma \in \Sigma$, then $\mathscr{S}_{\Sigma,B}(F)$ is flat over R.

Proof. Clearly, $\mathscr{S}_{\Sigma,B}(F)$ is flat over R if and only if $\mathscr{S}_\sigma(F_{(\sigma)}) = \mathscr{S}_{\Sigma,B}(F) \upharpoonright_{Y_\sigma}$ is flat over R for every $\sigma \in \Sigma$. By [ÉGA, IV.2.1.2] this is the case if and only if $F_{(\sigma)}$ is flat over R for every $\sigma \in \Sigma$, and thus a) is proven. Now, b) follows from a) and [AC, I.2.3 Proposition 2]. □

(3.1.17) Corollary *Suppose that Σ is full and simplicial, and let F be a P-graded $S_{(P)}$-module. Then, $\mathscr{S}_{\Sigma,P}(F)$ is flat over R if and only if F_σ is flat over R for every $\sigma \in \Sigma$.*

Proof. Suppose that $\mathscr{S}_{\Sigma,P}(F)$ is flat over R. Then, 3.1.16 a) implies that $F_{(\sigma)}$ is flat over R for every $\sigma \in \Sigma$. Moreover, by 3.1.15 we have $(F_\sigma)_\alpha \cong F_{(\sigma)}$ in $\mathsf{Mod}(S_{(\sigma)})$ for every $\sigma \in \Sigma$ and every $\alpha \in P$, and hence $(F_\sigma)_\alpha$ is flat over R for every $\sigma \in \Sigma$ and every $\alpha \in P$. Thus, $F_\sigma = \bigoplus_{\alpha \in P}(F_\sigma)_\alpha$ is flat over R by [AC, I.2.3 Proposition 2], and together with 3.1.16 b) the claim is proven. □

(3.1.18) We denote by $\mathbb{J}_{\Sigma,B}(R)$ the set of graded ideals of $S_{(B)}$, and we denote by $\widetilde{\mathbb{J}}_\Sigma(R)$ the set of quasicoherent ideals of $\mathscr{O}_{Y_\Sigma}$. On use of 3.1.6 and 3.1.7 we get a map

$$\Xi_{\Sigma,B,R} : \mathbb{J}_{\Sigma,B}(R) \to \widetilde{\mathbb{J}}_\Sigma(R),\ \mathfrak{a} \mapsto \mathscr{S}_{\Sigma,B}(\mathfrak{a}),$$

and by 3.1.5 it is clear that the diagram

$$\begin{tikzcd}
\mathbb{J}_{\Sigma,A}(R) \arrow[r, "\Xi_{\Sigma,A,R}"] \arrow[d] & \widetilde{\mathbb{J}}_{\Sigma}(R) \\
\mathbb{J}_{\Sigma,B}(R) \arrow[ur, "\Xi_{\Sigma,B,R}"'] &
\end{tikzcd}$$

in Ens, where the unmarked morphism is induced by the functor $\bullet_{(B)}$, commutes.

If no confusion can arise, then we set $\mathbb{J}_{\Sigma,B} := \mathbb{J}_{\Sigma,B}(R)$, $\widetilde{\mathbb{J}}_{\Sigma} := \widetilde{\mathbb{J}}_{\Sigma}(R)$, and $\Xi_{\Sigma,B} := \Xi_{\Sigma,B,R}$.

3.2. The first total functor of sections

This section is devoted to the preparations for proving that the functors $\mathscr{S}_{\Sigma,B}$ are essentially surjective. Keeping in mind the proof of the analogous statement for projective schemes (see [ÉGA, II.2.7.7]) we start by introducing a "total functor of sections". Since shifting and twisting are not necessarily compatible, there are two possibilities of defining such a functor. We consider only the first one here, resulting in the *first* total functor of sections. The second one – and the connections between the first and the second one – are studied later in Section 4.1.

We start with a construction that generalises [ÉGA, 0.5.4.6].

(3.2.1) Let $(X, \mathscr{O}_X)$ be a ringed space, let G be a group, and let

$$\mathbb{E} = \big((\mathscr{E}_g)_{g\in G}, (\mu_{g,h})_{(g,h)\in G^2}\big)$$

be a couple consisting of a family $(\mathscr{E}_g)_{g\in G}$ of $\mathscr{O}_X$-modules with $\mathscr{E}_0 = \mathscr{O}_X$ and of a family $(\mu_{g,h} : \mathscr{E}_g \otimes_{\mathscr{O}_X} \mathscr{E}_h \to \mathscr{E}_{g+h})_{(g,h)\in G^2}$ of morphisms in $\mathsf{Mod}(\mathscr{O}_X)$. Furthermore, we suppose that for all $g, h \in G$ it holds $\mu_{g,h} = \mu_{h,g} \circ \sigma_{g,h}$, where $\sigma_{g,h} : \mathscr{E}_g \otimes_{\mathscr{O}_X} \mathscr{E}_h \xrightarrow{\cong} \mathscr{E}_h \otimes_{\mathscr{O}_X} \mathscr{E}_g$ denotes the canonical isomorphism in $\mathsf{Mod}(\mathscr{O}_X)$, and that for all $g, h, k \in G$ it holds

$$\mu_{g+h,k} \circ (\mu_{g,h} \otimes_{\mathscr{O}_X} \mathrm{Id}_{\mathscr{E}_k}) = \mu_{g,h+k} \circ (\mathrm{Id}_{\mathscr{E}_g} \otimes_{\mathscr{O}_X} \mu_{h,k}).$$

Let $U \subseteq X$ be an open subset. We consider the G-graded $\mathscr{O}_X(U)^{(G)}$-module $\bigoplus_{g\in G} \Gamma(U, \mathscr{E}_g)$. For $g, h \in G$ we have a canonical morphism

$$\mathscr{E}_g(U) \otimes_{\mathscr{O}_X(U)} \mathscr{E}_h(U) \to (\mathscr{E}_g \otimes_{\mathscr{O}_X} \mathscr{E}_h)(U),$$

and it is readily checked that its composition with

$$\mu_{g,h}(U) : (\mathscr{E}_g \otimes_{\mathscr{O}_X} \mathscr{E}_h)(U) \to \mathscr{E}_{g+h}(U)$$

yields a structure of G-graded $\mathscr{O}_X(U)^{(G)}$-algebra on $\bigoplus_{g\in G} \Gamma(U, \mathscr{E}_g)$. This G-graded $\mathscr{O}_X(U)^{(G)}$-algebra is denoted by $\Gamma_*^{\mathbb{E},U}$. For every open subset $V \subseteq U$ the restriction morphisms $\mathscr{E}_g(U) \to \mathscr{E}_g(V)$ for $g \in G$ induce a morphism $\Gamma_*^{\mathbb{E},U} \to \Gamma_*^{\mathbb{E},V}$ of G-graded $\mathscr{O}_X(U)^{(G)}$-algebras, by abuse of language called *the restriction morphism* and denoted by $\upharpoonright_V$.

(3.2.2) We keep the hypotheses of 3.2.1. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We consider the G-graded $\mathscr{O}_X(U)^{(G)}$-module $\bigoplus_{g\in G}\Gamma(U,\mathscr{E}_g\otimes_{\mathscr{O}_X}\mathscr{F})$. For $g,h\in G$ we have a canonical morphism

$$\mathscr{E}_g(U)\otimes_{\mathscr{O}_X(U)}(\mathscr{E}_h\otimes_{\mathscr{O}_X}\mathscr{F})(U)\to(\mathscr{E}_g\otimes_{\mathscr{O}_X}\mathscr{E}_h\otimes_{\mathscr{O}_X}\mathscr{F})(U),$$

and it is readily checked that its composition with

$$(\mu_{g,h}\otimes_{\mathscr{O}_X}\mathrm{Id}_{\mathscr{F}})(U):(\mathscr{E}_g\otimes_{\mathscr{O}_X}\mathscr{E}_h\otimes_{\mathscr{O}_X}\mathscr{F})(U)\to(\mathscr{E}_{g+h}\otimes_{\mathscr{O}_X}\mathscr{F})(U)$$

yields a structure of G-graded $\Gamma_*^{\mathbb{E},U}$-module on $\bigoplus_{g\in G}\Gamma(U,\mathscr{E}_g\otimes_{\mathscr{O}_X}\mathscr{F})$. This G-graded $\Gamma_*^{\mathbb{E},U}$-module is denoted by $\Gamma_*^{\mathbb{E},U}(\mathscr{F})$. The above gives rise to a functor

$$\Gamma_*^{\mathbb{E},U}(\bullet):\mathsf{Mod}(\mathscr{O}_X)\to\mathsf{GrMod}^G(\Gamma_*^{\mathbb{E},U}).$$

For every open subset $V\subseteq U$ the restriction morphisms

$$(\mathscr{E}_g\otimes_{\mathscr{O}_X}\mathscr{F})(U)\to(\mathscr{E}_g\otimes_{\mathscr{O}_X}\mathscr{F})(V)$$

for $g\in G$ induce a morphism $\Gamma_*^{\mathbb{E},U}(\mathscr{F})\to\Gamma_*^{\mathbb{E},V}(\mathscr{F})$ of G-graded $\Gamma_*^{\mathbb{E},U}$-modules, by abuse of language called *the restriction morphism* and denoted by $\upharpoonright_V$. The above gives rise to a morphism

$$\Gamma_*^{\mathbb{E},U}\to\Gamma_*^{\mathbb{E},V}$$

in $\mathsf{Hom}(\mathsf{Mod}(\mathscr{O}_X),\mathsf{GrMod}^G(\Gamma_*^{\mathbb{E},U}))$, by abuse of language also called *the restriction morphism* and denoted by $\upharpoonright_V$.

The G-graded $\Gamma_*^{\mathbb{E},U}$-module underlying $\Gamma_*^{\mathbb{E},U}$ is $\Gamma_*^{\mathbb{E},U}(\mathscr{O}_X)$. If no confusion can arise, then we denote the G-graded $\mathscr{O}_X(U)^{(G)}$-algebra $\Gamma_*^{\mathbb{E},U}$ also by $\Gamma_*^{\mathbb{E},U}(\mathscr{O}_X)$.

Now we apply the above to Cox rings and Cox schemes to get the first total functor of sections.

(3.2.3) Let $\alpha,\beta\in A$. For every $\sigma\in\Sigma$, multiplication of S defines a monomorphism

$$(S_\sigma)_\alpha\otimes_{S_{(\sigma)}}(S_\sigma)_\beta\to(S_\sigma)_{\alpha+\beta}$$

in $\mathsf{Mod}(S_{(\sigma)})$ such that for all $\sigma,\tau\in\Sigma$ with $\tau\preccurlyeq\sigma$ the diagram

$$\begin{tikzcd}
(S_\sigma)_\alpha\otimes_{S_{(\sigma)}}(S_\sigma)_\beta \arrow[r] \arrow[d, "(\eta^\sigma_\tau)_\alpha\otimes(\eta^\sigma_\tau)_\beta"'] & (S_\sigma)_{\alpha+\beta} \arrow[d, "(\eta^\sigma_\tau)_{\alpha+\beta}"] \\
(S_\tau)_\alpha\otimes_{S_{(\tau)}}(S_\tau)_\beta \arrow[r] & (S_\tau)_{\alpha+\beta}
\end{tikzcd}$$

in $\mathsf{Mod}(S_{(\sigma)})$ commutes. Hence, by 3.1.2 and 3.1.3 we have for every $\sigma\in\Sigma$ a monomorphism

$$\mathscr{O}_{Y_\Sigma}(\alpha)(Y_\sigma)\otimes_{\mathscr{O}_{Y_\Sigma}(Y_\sigma)}\mathscr{O}_{Y_\Sigma}(\beta)(Y_\sigma)\to\mathscr{O}_{Y_\Sigma}(\alpha+\beta)(Y_\sigma)$$

in $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma}(Y_\sigma))$ such that the diagram

$$\begin{tikzcd}
\mathscr{O}_{Y_\Sigma}(\alpha)(Y_\sigma) \otimes_{\mathscr{O}_{Y_\Sigma}(Y_\sigma)} \mathscr{O}_{Y_\Sigma}(\beta)(Y_\sigma) \arrow[r] \arrow[d, "\upharpoonright_{Y_\tau}"'] & \mathscr{O}_{Y_\Sigma}(\alpha+\beta)(Y_\sigma) \arrow[d, "\upharpoonright_{Y_\tau}"] \\
\mathscr{O}_{Y_\Sigma}(\alpha)(Y_\tau) \otimes_{\mathscr{O}_{Y_\Sigma}(Y_\tau)} \mathscr{O}_{Y_\Sigma}(\beta)(Y_\tau) \arrow[r] & \mathscr{O}_{Y_\Sigma}(\alpha+\beta)(Y_\tau)
\end{tikzcd}$$

in $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma}(Y_\sigma))$ commutes. So, by glueing these morphisms give rise to a monomorphism

$$\mu^{\Sigma}_{\alpha,\beta} := \mu^{\Sigma,R}_{\alpha,\beta} : \mathscr{O}_{Y_\Sigma}(\alpha) \otimes_{\mathscr{O}_{Y_\Sigma}} \mathscr{O}_{Y_\Sigma}(\beta) \to \mathscr{O}_{Y_\Sigma}(\alpha+\beta).$$

If Σ is full and $\alpha, \beta \in P$, then 3.1.13 implies that $\mu^{\Sigma}_{\alpha,\beta}$ is an isomorphism.

(3.2.4) On use of associativity and commutativity of S_σ for $\sigma \in \Sigma$ it is readily checked that the family

$$\mathbb{E}_{\Sigma,R} := \big((\mathscr{O}_{Y_\Sigma}(\alpha))_{\alpha\in A}, (\mu^{\Sigma}_{\alpha,\beta})_{(\alpha,\beta)\in A^2}\big)$$

fulfils the hypotheses of 3.2.1. Hence, it defines for every open subset $U \subseteq Y_\Sigma$ an A-graded $\mathscr{O}_{Y_\Sigma}(U)^{(A)}$-algebra $\Gamma^{\Sigma,R,U}_*(\mathscr{O}_{Y_\Sigma}) := \Gamma^{\mathbb{E}_{\Sigma,R},U}_*(\mathscr{O}_{Y_\Sigma})$, and a functor

$$\Gamma^{\Sigma,R,U}_* := \Gamma^{\mathbb{E}_{\Sigma,R},U}_* : \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}) \to \mathsf{GrMod}^A(\Gamma^{\Sigma,R,U}_*(\mathscr{O}_{Y_\Sigma})),$$

called *the first total functor of sections over U associated with Σ over R.* If no confusion can arise, then we set $\Gamma^{\Sigma,U}_* := \Gamma^{\Sigma,R,U}_*$, $\Gamma^{\Sigma}_* := \Gamma^{\Sigma,R,Y_\Sigma}_*$ and $\Gamma^{\sigma}_* := \Gamma^{\Sigma,R,Y_\sigma}_*$ for $\sigma \in \Sigma$.

For all open subsets $U, V \subseteq Y_\Sigma$ with $V \subseteq U$ we have the restriction morphisms

$$\Gamma^{\Sigma,U}_*(\mathscr{O}_{Y_\Sigma}) \to \Gamma^{\Sigma,V}_*(\mathscr{O}_{Y_\Sigma})$$

in $\mathsf{GrAlg}^A(\mathscr{O}_{Y_\Sigma}(U)^{(A)})$ and

$$\Gamma^{\Sigma,U}_* \to \Gamma^{\Sigma,V}_*$$

in $\mathsf{Hom}(\mathsf{Mod}(\mathscr{O}_{Y_\Sigma}), \mathsf{GrMod}^A(\Gamma^{\Sigma,U}_*(\mathscr{O}_{Y_\Sigma})))$. In particular, for every $\sigma \in \Sigma$ we have the restriction morphisms

$$\Gamma^{\Sigma}_*(\mathscr{O}_{Y_\Sigma}) \to \Gamma^{\sigma}_*(\mathscr{O}_{Y_\Sigma})$$

in $\mathsf{GrAlg}^A(\mathscr{O}_{Y_\Sigma}(Y_\Sigma)^{(A)})$ and

$$\Gamma^{\Sigma}_* \to \Gamma^{\sigma}_*$$

in $\mathsf{Hom}(\mathsf{Mod}(\mathscr{O}_{Y_\Sigma}), \mathsf{GrMod}^A(\Gamma^{\Sigma}_*(\mathscr{O}_{Y_\Sigma})))$, and for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ we have the restriction morphisms

$$\Gamma^{\sigma}_*(\mathscr{O}_{Y_\Sigma}) \to \Gamma^{\tau}_*(\mathscr{O}_{Y_\Sigma})$$

in $\mathsf{GrAlg}^A(\mathscr{O}_{Y_\Sigma}(Y_\sigma)^{(A)})$ and

$$\Gamma^{\sigma}_* \to \Gamma^{\tau}_*$$

in $\mathsf{Hom}(\mathsf{Mod}(\mathscr{O}_{Y_\Sigma}), \mathsf{GrMod}^A(\Gamma^{\sigma}_*(\mathscr{O}_{Y_\Sigma})))$.

(3.2.5) For every $\alpha \in A$ and all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$, we have the commutative diagram

$$\begin{tikzcd} S_\alpha \arrow[r, "(\eta_\sigma)_\alpha"] \arrow[dr, "(\eta_\tau)_\alpha"'] & \mathscr{O}_{Y_\Sigma}(\alpha)(Y_\sigma) \arrow[d, "\upharpoonright_{Y_\tau}"] \\ & \mathscr{O}_{Y_\Sigma}(\alpha)(Y_\tau) \end{tikzcd}$$

in $\mathsf{Mod}(S_0)$. Hence, there is a unique morphism

$$\eta_{\Sigma,\alpha} : S_\alpha \to \mathscr{O}_{Y_\Sigma}(\alpha)(Y_\Sigma)$$

in $\mathsf{Mod}(S_0)$ such that for every $\sigma \in \Sigma$ the diagram

$$\begin{tikzcd} S_\alpha \arrow[r, "\eta_{\Sigma,\alpha}"] \arrow[dr, "(\eta_\sigma)_\alpha"'] & \mathscr{O}_{Y_\Sigma}(\alpha)(Y_\Sigma) \arrow[d, "\upharpoonright_{Y_\sigma}"] \\ & \mathscr{O}_{Y_\Sigma}(\alpha)(Y_\sigma) \end{tikzcd}$$

in $\mathsf{Mod}(S_0)$ commutes. So, we get a morphism

$$\eta_\Sigma := \bigoplus_{\alpha \in A} \eta_{\Sigma,\alpha} : S \to \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})$$

in $\mathsf{Mod}(S_0)$, and it is readily checked that this is a morphism in GrAnn^A. By means of this we consider $\Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})$ as an A-graded S-algebra. Thus, by composition with scalar restriction by means of η_Σ we can and do consider Γ_*^Σ as a functor from $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma})$ to $\mathsf{GrMod}^A(S)$.

(3.2.6) Proposition *The morphism $\eta_\Sigma : S \to \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})$ in GrAnn^A is surjective, and it is an isomorphism if and only if $\Sigma \neq \emptyset$ or $R = 0$.*

Proof. Clearly, it suffices to show that for every $\alpha \in A$ the morphism $\eta_{\Sigma,\alpha}$ is surjective, and an isomorphism if and only if $\Sigma \neq \emptyset$ or $R = 0$. So, let $\alpha \in A$.

First, let $x \in \mathscr{O}_{Y_\Sigma}(\alpha)(Y_\Sigma)$. For every $\sigma \in \Sigma$ we have $x \upharpoonright_{Y_\sigma} \in S(\alpha)_{(\sigma)} = (S_\sigma)_\alpha$, and hence there exists an $m \in \mathbb{N}_0$ such that for every $\sigma \in \Sigma$ there exists a $y^{(\sigma)} \in S_{\alpha + m\widehat{\alpha}_\sigma}$ such that $x \upharpoonright_{Y_\sigma} = \frac{y^{(\sigma)}}{\widehat{Z}_\sigma^m}$. For every $\sigma \in \Sigma$ it holds

$$\frac{y^{(\sigma)} \cdot \prod_{\rho \in \sigma_1} Z_\rho^m}{\prod_{\rho \in \Sigma_1} Z_\rho^m} = x \upharpoonright_{Y_\sigma} \upharpoonright_{Y_{\{0\}}} = x \upharpoonright_{Y_{\{0\}}} = \frac{y^{(\{0\})}}{\prod_{\rho \in \Sigma_1} Z_\rho^m} \in (S_{\{0\}})_\alpha$$

and hence $y^{(\{0\})} = y^{(\sigma)} \cdot \prod_{\rho \in \sigma_1} Z_\rho^m$. Therefore, Z_ρ^m divides $y^{(\{0\})}$ for every $\rho \in \Sigma_1$, and thus there exists $y \in S_\alpha$ such that $\frac{y}{1} = x \upharpoonright_{Y_{\{0\}}} \in (S_{\{0\}})_\alpha$. Now, for every $\sigma \in \Sigma$ we get

$$\eta_{\Sigma,\alpha}(y) \upharpoonright_{Y_\sigma} = \frac{y}{1} = \frac{y^{(\sigma)}}{\widehat{Z}_\sigma^m} = x \upharpoonright_{Y_\sigma} \in (S_\sigma)_\alpha,$$

and this implies $\eta_{\Sigma,\alpha}(y) = x$. Thus, $\eta_{\Sigma,\alpha}$ is surjective.

Now, we show the second statement. If $\Sigma = \emptyset$, then $\eta_{\Sigma,\alpha}$ is a morphism of rings $R \to 0$ and hence an isomorphism if and only if $R = 0$. So, let $\Sigma \neq \emptyset$, and let $x \in \mathrm{Ker}(\eta_\alpha)$. For every $\sigma \in \Sigma$ it holds $\frac{x}{1} = \eta_\alpha(x) \upharpoonright_{Y_\sigma} = 0 \in (S_\sigma)_\alpha$, and hence there exists an $m \in \mathbb{N}_0$ such that $\widehat{Z}_\sigma^m x = 0 \in S$. But from this we get $x = 0$, and hence $\eta_{\Sigma,\alpha}$ is injective. Herewith, the claim is proven. $\square$

(3.2.7) For every $\sigma \in \Sigma$ it holds $S_\sigma = \Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma})$ by definition of $\Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma})$, and for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
S \arrow[rr, "\eta_\sigma"] \arrow[dr, "\eta_\tau"] \arrow[dd, "\eta_\Sigma"'] & & S_\sigma \arrow[dl, "\eta_\tau^\sigma"'] \arrow[dd, "(\eta_\Sigma)_\sigma"] \\
& S_\tau \arrow[dd, "(\eta_\Sigma)_\tau"] & \\
\Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma}) \arrow[rr, "\upharpoonright_{Y_\sigma}"] \arrow[dr, "\upharpoonright_{Y_\tau}"'] & & \Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma}) \arrow[dl, "\upharpoonright_{Y_\tau}"] \\
& \Gamma_*^\tau(\mathscr{O}_{Y_\Sigma}) &
\end{tikzcd}$$

in $\mathsf{GrAlg}^A(S)$ commutes by 3.2.6. Therefore, η_Σ induces for every $\sigma \in \Sigma$ an isomorphism

$$(\eta_\Sigma)_\sigma^{-1} : \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})_\sigma \xrightarrow{\cong} \Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma})$$

in $\mathsf{GrAlg}^A(S)$ such that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
& \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma}) \arrow[d, "\eta_\sigma"] \arrow[ddl, "\eta_\tau"'] \arrow[drr, "\upharpoonright_{Y_\sigma}"] \arrow[ddr, "\upharpoonright_{Y_\tau}"] & & \\
& \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})_\sigma \arrow[rr, "\cong"'] \arrow[dl, "\eta_\tau^\sigma"] & & \Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma}) \arrow[dl, "\upharpoonright_{Y_\tau}"] \\
\Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})_\tau \arrow[rr, "\cong"'] & & \Gamma_*^\tau(\mathscr{O}_{Y_\Sigma}) &
\end{tikzcd}$$

in $\mathsf{GrAlg}^A(S)$ commutes. By means of these isomorphisms we identify for every $\sigma \in \Sigma$ the A-graded $\Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})$-algebras $\upharpoonright_{Y_\sigma} : \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma}) \to \Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma})$ and $\eta_\sigma : \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma}) \to \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})_\sigma$. Hence, for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the morphisms $\upharpoonright_{Y_\tau} : \Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma}) \to \Gamma_*^\tau(\mathscr{O}_{Y_\Sigma})$ and $\eta_\tau^\sigma : \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})_\sigma \to \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})_\tau$ in $\mathsf{GrAlg}^A(\Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma}))$ are identified, too.

3.3. On surjectivity of $\mathscr{S}_\Sigma$

In this section we prove that the functors $\mathscr{S}_{\Sigma,B}$ are essentially surjective. The proof is based on [18], where Weil divisor techniques lead to the corresponding result for toric varieties, and on the proof of the projective analogue in [ÉGA, II.2.7.7].

(3.3.1) Proposition *Let $\sigma, \tau \in \Sigma$. Then, there exist $p \in S_{(\tau)}$, $l \in \mathbb{N}$, and $q \in (S_\tau)_{l\widehat{\alpha}_\sigma}$ such that*[4] *$Y_{\tau\cap\sigma} = (Y_\tau)_p$ and that $qp = \widehat{Z}_\sigma^l \in (S_\tau)_{l\widehat{\alpha}_\sigma}$.*

Proof. If $\tau \preccurlyeq \sigma$, then $p = 1$, $l = 1$ and $q = \widehat{Z}_\sigma$ obviously fulfil the claim. So, let $\tau \not\preccurlyeq \sigma$. As $\tau \cap \sigma \preccurlyeq \tau$, there is a $u \in \tau_M^\vee$ such that $\tau \cap \sigma = \tau \cap u^\perp$.

[4] using the notations from [ÉGA, 0.4.1.9]

Then, we have $p := \varepsilon_{(\tau)}(e_{c(u)}) \in S_{(\tau)}$. As $\tau \not\preccurlyeq \sigma$ it holds $\tau_1 \setminus \sigma_1 \neq \emptyset$, and hence $l := \max\{\rho_N(u) \mid \rho \in \tau_1 \setminus \sigma_1\} \in \mathbb{N}$ exists. Furthermore, we have

$$q := (\textstyle\prod_{\rho\in\Sigma_1\setminus(\tau_1\cup\sigma_1)} Z_\rho^l)(\prod_{\rho\in\tau_1\setminus\sigma_1} Z_\rho^{l-\rho_N(u)})(\prod_{\rho\in\Sigma_1\setminus\tau_1} Z_\rho^{-\rho_N(u)}) \in S_\tau$$

and $qp = \widehat{Z}_\sigma^l \in (S_\tau)_{l\widehat{\alpha}_\sigma}$, and hence we get $q \in (S_\tau)_{l\widehat{\alpha}_\sigma}$. Finally, from 2.2.8 and 2.2.1 we see that the canonical injection $\kappa_{\tau\cap\sigma,\tau} : Y_{\tau\cap\sigma} \hookrightarrow Y_\tau$ is the spectrum of the canonical morphism $\eta_p : S_{(\tau)} \to (S_{(\tau)})_p$, and thus it holds $Y_{\tau\cap\sigma} = (Y_\tau)_p$. □

(3.3.2) Let $\mathscr{F}$ be an $\mathscr{O}_{Y_\Sigma}$-module, let $\sigma \in \Sigma$, and let $f \in \Gamma_*^\Sigma(\mathscr{F})_{(\sigma)}$. Then, there are $m \in \mathbb{N}_0$ and $x \in \Gamma_*^\Sigma(\mathscr{F})_{m\widehat{\alpha}_\sigma}$ such that $f = \frac{x}{\widehat{Z}_\sigma^m}$. So, it holds $x \restriction_{Y_\sigma} \in \Gamma_*^\sigma(\mathscr{F})_{m\widehat{\alpha}_\sigma}$ and $\frac{1}{\widehat{Z}_\sigma^m} \restriction_{Y_\sigma} = \frac{1}{\widehat{Z}_\sigma^m} \in \Gamma_*^\sigma(\mathscr{O}_{Y_\Sigma})_{-m\widehat{\alpha}_\sigma}$, and hence we get $\frac{1}{\widehat{Z}_\sigma^m} \cdot x \restriction_{Y_\sigma} \in \Gamma_*^\sigma(\mathscr{F})_0 = \mathscr{F}(Y_\sigma)$. In this way we get a morphism

$$\beta_\sigma(\mathscr{F}) : \Gamma_*^\Sigma(\mathscr{F})_{(\sigma)} \to \mathscr{F}(Y_\sigma),\ \frac{x}{\widehat{Z}_\sigma^m} \mapsto \frac{1}{\widehat{Z}_\sigma^m} \cdot x \restriction_{Y_\sigma}$$

in $\mathsf{Mod}_{S_{(\sigma)}}$. It is readily checked that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
\Gamma_*^\Sigma(\mathscr{F})_{(\sigma)} \arrow[r, "\beta_\sigma(\mathscr{F})"] \arrow[d, "\eta_{(\tau)}^{(\sigma)}"'] & \mathscr{F}(Y_\sigma) \arrow[d, "\restriction_{Y_\tau}"] \\
\Gamma_*^\Sigma(\mathscr{F})_{(\tau)} \arrow[r, "\beta_\tau(\mathscr{F})"] & \mathscr{F}(Y_\tau)
\end{tikzcd}$$

in $\mathsf{Mod}(S_{(\sigma)})$ commutes. Thus, there is a unique morphism

$$\beta_\Sigma(\mathscr{F}) : \mathscr{S}_\Sigma(\Gamma_*^\Sigma(\mathscr{F})) \to \mathscr{F}$$

in $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma})$ with $\beta_\Sigma(\mathscr{F})(Y_\sigma) = \beta_\sigma(\mathscr{F})$ for every $\sigma \in \Sigma$.

Varying $\mathscr{F}$ in the above yields a morphism

$$\beta_\Sigma : \mathscr{S}_\Sigma \circ \Gamma_*^\Sigma \to \mathrm{Id}_{\mathsf{Mod}(\mathscr{O}_{Y_\Sigma})}$$

in $\mathsf{Hom}(\mathsf{Mod}(\mathscr{O}_{Y_\Sigma}), \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}))$. It follows from 3.1.4 that by means of restriction and coastriction β_Σ induces a morphism

$$\mathscr{S}_\Sigma \circ \Gamma_*^\Sigma \restriction_{\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma})} \to \mathrm{Id}_{\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma})}$$

in $\mathsf{Hom}(\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma}), \mathsf{QCMod}(\mathscr{O}_{Y_\Sigma}))$ that we also denote by β_Σ.

(3.3.3) Proposition *Let $\sigma \in \Sigma$, and let $\mathscr{F}$ be a quasicoherent $\mathscr{O}_{Y_\Sigma}$-module. Then, the morphism*

$$\beta_\sigma(\mathscr{F}) : \Gamma_*^\Sigma(\mathscr{F})_{(\sigma)} \to \mathscr{F}(Y_\sigma)$$

in $\mathsf{Mod}(S_{(\sigma)})$ is surjective.

Proof. Let $f \in \mathscr{F}(Y_\sigma)$. We construct a preimage of f under $\beta_\sigma(\mathscr{F})$. Let $\tau \in \Sigma$. By 3.3.1 there are $p_\tau \in S_{(\tau)}$, $j_\tau \in \mathbb{N}$ and $q_\tau \in (S_\tau)_{j_\tau\widehat{\alpha}_\sigma}$ such that $Y_{\tau\cap\sigma} = (Y_\tau)_{p_\tau}$ and that $q_\tau p_\tau = \widehat{Z}_\sigma^{j_\tau} \in (S_\tau)_{j_\tau\widehat{\alpha}_\sigma}$. Hence we have the affine scheme Y_τ, the invertible $\mathscr{O}_{Y_\tau}$-module $\mathscr{O}_{Y_\tau}$, the quasicoherent $\mathscr{O}_{Y_\tau}$-module $\mathscr{F} \restriction_{Y_\tau}$, and the sections $p_\tau \in S_{(\tau)} = \mathscr{O}_{Y_\tau}(Y_\tau)$ and $f \restriction_{Y_{\tau\cap\sigma}} \in \mathscr{F}(Y_{\tau\cap\sigma}) =$

$\mathscr{F}((Y_\tau)_{p_\tau})$. Therefore, [ÉGA, I.6.8.1] implies the existence of $k_\tau \in \mathbb{N}$ and $g''_\tau \in \mathscr{F}(Y_\tau)$ such that

$$g''_\tau \restriction_{Y_{\tau\cap\sigma}} = p_\tau^{k_\tau} \restriction_{Y_{\tau\cap\sigma}} f \restriction_{Y_{\tau\cap\sigma}} \in \mathscr{F}(Y_{\tau\cap\sigma}),$$

and we set $k := \max\{k_\tau | \tau \in \Sigma\} \in \mathbb{N}$ and $j := \max\{j_\tau | \tau \in \Sigma\} \in \mathbb{N}$. Then, for every $\tau \in \Sigma$ it holds

$$g'_\tau := \widehat{Z}_\sigma^{k(j-j_\tau)} q_\tau^k p_\tau^{k-k_\tau} g''_\tau \in \mathscr{F}(kj\widehat{\alpha}_\sigma)(Y_\tau)$$

and

$$g'_\tau \restriction_{Y_{\tau\cap\sigma}} = (\widehat{Z}_\sigma^{k(j-j_\tau)} q_\tau^k p_\tau^{k-k_\tau} g''_\tau) \restriction_{Y_{\tau\cap\sigma}} =$$
$$(\widehat{Z}_\sigma^{k(j-j_\tau)} q_\tau^k p_\tau^{k-k_\tau}) \restriction_{Y_{\tau\cap\sigma}} p_\tau^{k_\tau} \restriction_{Y_{\tau\cap\sigma}} f \restriction_{Y_{\tau\cap\sigma}} =$$
$$(\widehat{Z}_\sigma^{k(j-j_\tau)} (q_\tau p_\tau)^k) \restriction_{Y_{\tau\cap\sigma}} f \restriction_{Y_{\tau\cap\sigma}} = \widehat{Z}_\sigma^{kj} f \restriction_{Y_{\tau\cap\sigma}} \in \mathscr{F}(kj\widehat{\alpha}_\sigma)(Y_{\tau\cap\sigma}).$$

Now, let $\tau, \tau' \in \Sigma$. We set $\omega := \tau \cap \tau'$ and

$$h_{\tau\tau'} := g'_\tau \restriction_{Y_\omega} - g'_{\tau'} \restriction_{Y_\omega} \in \mathscr{F}(kj\widehat{\alpha}_\sigma)(Y_\omega).$$

By 3.3.1 there are $r_{\tau\tau'} \in S_{(\omega)}$, $l_{\tau\tau'} \in \mathbb{N}$ and $s_{\tau\tau'} \in (S_\omega)_{l_{\tau\tau'}\widehat{\alpha}_\sigma}$ such that $Y_{\omega\cap\sigma} = (Y_\omega)_{r_{\tau\tau'}}$ and that $s_{\tau\tau'} r_{\tau\tau'} = \widehat{Z}_\sigma^{l_{\tau\tau'}}$. Hence we have the affine scheme Y_ω, the invertible $\mathscr{O}_{Y_\omega}$-module $\mathscr{O}_{Y_\omega}$, the quasicoherent $\mathscr{O}_{Y_\omega}$-module $\mathscr{F} \restriction_{Y_\omega}$, and the sections $r_{\tau\tau'} \in S_{(\omega)} = \mathscr{O}_{Y_\omega}(Y_\omega)$ and $h_{\tau\tau'} \in \mathscr{F}(kj\widehat{\alpha}_\sigma)(Y_\omega)$ with

$$h_{\tau\tau'} \restriction_{(Y_\omega)_{r_{\tau\tau'}}} = h_{\tau\tau'} \restriction_{Y_{\omega\cap\sigma}} = g'_\tau \restriction_{Y_\omega} \restriction_{Y_{\omega\cap\sigma}} - g'_{\tau'} \restriction_{Y_\omega} \restriction_{Y_{\omega\cap\sigma}} =$$

$$g'_\tau \restriction_{\tau\cap\sigma} \restriction_{Y_{\tau\cap\sigma\cap\tau'}} - g'_{\tau'} \restriction_{\tau'\cap\sigma} \restriction_{Y_{\tau'\cap\sigma\cap\tau}} = \widehat{Z}_\sigma^{kj} f \restriction_{Y_{\tau\cap\sigma\cap\tau'}} - \widehat{Z}_\sigma^{kj} f \restriction_{Y_{\tau'\cap\sigma\cap\tau}} = 0.$$

Therefore, [ÉGA, I.6.8.1] implies the existence of $m_{\tau\tau'} \in \mathbb{N}$ such that $r_{\tau\tau'}^{m_{\tau\tau'}} h_{\tau\tau'} = 0 \in \mathscr{F}(kj\widehat{\alpha}_\sigma)(Y_\omega)$, and we set

$$l := \max\{l_{\tau\tau'} m_{\tau\tau'} | (\tau, \tau') \in \Sigma^2\} \in \mathbb{N}.$$

Then, for every $\tau \in \Sigma$ it holds

$$g_\tau := \widehat{Z}_\sigma^l g'_\tau \in \mathscr{F}((kj + l)\widehat{\alpha}_\sigma)(Y_\tau),$$

and if $\tau, \tau' \in \Sigma$ and $\omega := \tau \cap \tau'$, then it follows

$$g_\tau \restriction_{Y_\omega} - g_{\tau'} \restriction_{Y_\omega} = \widehat{Z}_\sigma^l g'_\tau \restriction_{Y_\omega} - \widehat{Z}_\sigma^l g'_{\tau'} \restriction_{Y_\omega} = \widehat{Z}_\sigma^{l-l_{\tau\tau'}m_{\tau\tau'}} \widehat{Z}_\sigma^{l_{\tau\tau'}m_{\tau\tau'}} h_{\tau\tau'} =$$
$$\widehat{Z}_\sigma^{l-l_{\tau\tau'}m_{\tau\tau'}} s_{\tau\tau'}^{m_{\tau\tau'}} r_{\tau\tau'}^{m_{\tau\tau'}} h_{\tau\tau'} = 0 \in \mathscr{F}((kj + l)\widehat{\alpha}_\sigma)(Y_\omega).$$

Thus, there exists $g \in \mathscr{F}((kj+l)\widehat{\alpha}_\sigma)(Y)$ such that $g \restriction_{Y_\tau} = g_\tau$ for every $\tau \in \Sigma$, and it holds $\frac{g}{\widehat{Z}_\sigma^{kj+l}} \in \Gamma_*^\Sigma(\mathscr{F})_{(\sigma)}$.

Finally, for every $\tau \in \Sigma$ we have

$$\frac{g\restriction_{Y_\sigma}}{\widehat{Z}_\sigma^{kj+l}} \restriction_{Y_{\tau\cap\sigma}} = \frac{g\restriction_{Y_{\tau\cap\sigma}}}{\widehat{Z}_\sigma^{kj+l}} = \frac{g_\tau\restriction_{Y_{\tau\cap\sigma}}}{\widehat{Z}_\sigma^{kj+l}} = \frac{\widehat{Z}_\sigma^l g'_\tau\restriction_{Y_{\tau\cap\sigma}}}{\widehat{Z}_\sigma^{kj+l}} = \frac{\widehat{Z}_\sigma^{kj+l} f\restriction_{Y_{\tau\cap\sigma}}}{\widehat{Z}_\sigma^{kj+l}} = f \restriction_{Y_{\tau\cap\sigma}},$$

and this implies $\beta_\sigma(\mathscr{F})(\frac{g}{\widehat{Z}_\sigma^{kj+l}}) = \frac{g\restriction_{Y_\sigma}}{\widehat{Z}_\sigma^{kj+l}} = f$ as desired. □

(3.3.4) Proposition *Let $\sigma \in \Sigma$, and let $\mathscr{F}$ be a quasicoherent $\mathscr{O}_{Y_\Sigma}$-module. Then, the morphism*

$$\beta_\sigma(\mathscr{F}) : \Gamma_*^\Sigma(\mathscr{F})_{(\sigma)} \to \mathscr{F}(Y_\sigma)$$

in $\mathsf{Mod}(S_{(\sigma)})$ is injective.

Proof. Let $f \in \Gamma_*^{\Sigma}(\mathscr{F})_{(\sigma)}$ be such that $\beta_\sigma(\mathscr{F})(f) = 0$. Then, there are $k \in \mathbb{N}_0$ and $g \in \mathscr{F}(k\widehat{\alpha}_\sigma)(Y)$ with $f = \frac{g}{\widehat{Z}_\sigma^k}$, and it holds $\frac{g\restriction_{Y_\sigma}}{\widehat{Z}_\sigma^k} = 0 \in \mathscr{F}(Y_\sigma)$, hence $g\restriction_{Y_\sigma} = \widehat{Z}_\sigma^l \frac{g\restriction_{Y_\sigma}}{\widehat{Z}_\sigma^l} = 0 \in \mathscr{F}(k\widehat{\alpha}_\sigma)(Y_\sigma)$. So, to show $f = 0$ we have to show the existence of $l \in \mathbb{N}_0$ with $\widehat{Z}_\sigma^l g = 0 \in \mathscr{F}((l+k)\widehat{\alpha}_\sigma)(Y)$.

Let $\tau \in \Sigma$. By 3.3.1 there are $p_\tau \in S_{(\tau)}$, $m_\tau \in \mathbb{N}$ and $q_\tau \in (S_\tau)_{m_\tau\delta}$ such that $Y_{\tau\cap\sigma} = (Y_\tau)_{p_\tau}$ and that $q_\tau p_\tau = \widehat{Z}_\sigma^{m_\tau} \in (S_\tau)_{m_\tau\widehat{\alpha}_\sigma}$. Hence we have the affine scheme Y_τ, the invertible $\mathscr{O}_{Y_\tau}$-module $\mathscr{O}_{Y_\tau}$, the quasicoherent $\mathscr{O}_{Y_\tau}$-module $\mathscr{F}(k\widehat{\alpha}_\sigma)\restriction_{Y_\tau}$, and the sections $p_\tau \in S_{(\tau)} = \mathscr{O}_{Y_\tau}(Y_\tau)$ and $g\restriction_{Y_\tau} \in \mathscr{F}(k\widehat{\alpha}_\sigma)(Y_\tau)$ with

$$g\restriction_{Y_\tau}\restriction_{(Y_\tau)_{p_\tau}} = g\restriction_{Y_\tau}\restriction_{Y_{\tau\cap\sigma}} = g\restriction_{Y_\sigma}\restriction_{Y_{\tau\cap\sigma}} = 0 \in \mathscr{F}(k\widehat{\alpha}_\sigma)((Y_\tau)_{p_\tau}).$$

Therefore, [ÉGA, I.6.8.1] implies the existence of $l_\tau \in \mathbb{N}$ such that

$$p_\tau^{l_\tau}(g\restriction_{Y_\tau}) = 0 \in \mathscr{F}(k\widehat{\alpha}_\sigma)(Y\tau).$$

From this we get

$$(\widehat{Z}_\sigma^{l_\tau m_\tau} g)\restriction_{Y_\tau} = q_\tau^{l_\tau} p_\tau^{l_\tau}(g\restriction_{Y_\tau}) = 0 \in \mathscr{F}((l_\tau m_\tau + k)\widehat{\alpha}_\sigma)(Y_\tau).$$

Setting $l := \max\{l_\tau m_\tau | \tau \in \Sigma\} \in \mathbb{N}$ it follows

$$(\widehat{Z}_\sigma^l g)\restriction_{Y_\tau} = 0 \in \mathscr{F}((l+k)\widehat{\alpha}_\sigma)(Y_\tau)$$

for every $\tau \in \Sigma$ and thus $\widehat{Z}_\sigma^l g = 0 \in \mathscr{F}((l+k)\widehat{\alpha}_\sigma)(Y)$ as desired. □

(3.3.5) Theorem *The morphism*

$$\beta_\Sigma : \mathscr{S}_\Sigma \circ \Gamma_*^\Sigma\restriction_{\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma})} \to \mathrm{Id}_{\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma})}$$

in $\mathsf{Hom}(\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma}), \mathsf{QCMod}(\mathscr{O}_{Y_\Sigma}))$ *is an isomorphism.*

Proof. If $\mathscr{F}$ is a quasicoherent $\mathscr{O}_{Y_\Sigma}$-module, then $\beta_\Sigma(\mathscr{F})(Y_\sigma) = \beta_\sigma(\mathscr{F})$ is an isomorphism for every $\sigma \in \Sigma$ by 3.3.3 and 3.3.4, and hence $\beta_\Sigma(\mathscr{F})$ is an isomorphism, too. The claim follows from this. □

(3.3.6) Corollary *Let $B \subseteq A$ be a big subgroup. Then, the functor*

$$\mathscr{S}_{\Sigma,B} : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{QCMod}(\mathscr{O}_{Y_\Sigma})$$

is essentially surjective.

Proof. If $\mathscr{F}$ is a quasicoherent $\mathscr{O}_{Y_\Sigma}$-module, then $\Gamma_*^\Sigma(\mathscr{F})$ is an A-graded S-module, and by 3.3.5 it holds $\mathscr{S}_\Sigma(\Gamma_*^\Sigma(\mathscr{F})) \cong \mathscr{F}$. Now, the claim follows from 3.1.5. □

As an applications of the above we derive corollaries on surjectivity of the functors $\mathscr{S}_{\Sigma,B}$ with respect to ideals and finiteness conditions, respectively.

(3.3.7) Corollary *Let $B \subseteq A$ be a big subgroup. Then, the map*

$$\Xi_{\Sigma,B} : \mathbb{J}_{\Sigma,B} \to \widetilde{\mathbb{J}}_\Sigma,\ \mathfrak{a} \mapsto \mathscr{S}_{\Sigma,B}(\mathfrak{a})$$

is surjective.

Proof. By 3.1.18 we can suppose without loss of generality that $B = A$. If $\Sigma = \emptyset$, then the claim is obvious. So, let $\Sigma \neq \emptyset$. Then, $\eta_\Sigma : S \to \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})$ is an isomorphism by 3.2.6. Let $\mathscr{J}$ be a quasicoherent ideal of $\mathscr{O}_{Y_\Sigma}$, and let $j : \mathscr{J} \hookrightarrow \mathscr{O}_{Y_\Sigma}$ denote the canonical injection. Considering the graded ideal $\mathfrak{a} := \mathrm{Im}(\eta_\Sigma^{-1} \circ \Gamma_*^\Sigma(j)) \subseteq S$ and applying $\mathscr{S}_\Sigma$ we get by 3.2.3 and 3.1.7 the diagram

$$\begin{tikzcd}
\mathscr{O}_{Y_\Sigma} & \mathscr{S}_\Sigma(\Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})) \arrow[l, "\beta_\Sigma(\mathscr{O}_{Y_\Sigma})"', "\cong"] \arrow[r, "\mathscr{S}_\Sigma(\eta_\Sigma^{-1})", "\cong"'] & \mathscr{S}_\Sigma(S) \\
\mathscr{J} \arrow[u, hook, "j"] & \mathscr{S}_\Sigma(\Gamma_*^\Sigma(\mathscr{J})) \arrow[l, "\beta_\Sigma(\mathscr{J})"', "\cong"] \arrow[u, "\mathscr{S}_\Sigma(\Gamma_*^\Sigma(j))"'] \arrow[r, two heads] & \mathscr{S}_\Sigma(\mathfrak{a}) \arrow[u, hook]
\end{tikzcd}$$

in $\mathsf{GrMod}^A(S)$, where the unmarked morphisms are the canonical ones, and it suffices to show that it commutes. The two quadrangles commute, and so it suffices to show that $\mathscr{S}_\Sigma(\eta_\Sigma)$ is the inverse of $\beta_\Sigma(\mathscr{O}_{Y_\Sigma})$. But this follows immediately from the definitions of these morphisms. □

(3.3.8) Corollary *Let $B \subseteq A$ be a big subgroup.*

a) If $\mathscr{F}$ is a quasicoherent $\mathscr{O}_{Y_\Sigma}$-module of finite type, then there exists a finitely generated B-graded $S_{(B)}$-module F such that $\mathscr{F} \cong \mathscr{S}_{\Sigma,B}(F)$.

b) If $\mathscr{J}$ is a quasicoherent ideal of $\mathscr{O}_{Y_\Sigma}$ of finite type, then there exists a finitely generated graded ideal $\mathfrak{a}$ of $S_{(B)}$ such that $\mathscr{F} \cong \mathscr{S}_{\Sigma,B}(\mathfrak{a})$.

Proof. We prove a) and b) simultaneously, and we write $\mathscr{F}$ instead of $\mathscr{J}$ in case b). By 3.3.6 and 3.3.7, respectively, there exists a B-graded $S_{(B)}$-module F, or a graded ideal $F \subseteq S_{(B)}$, respectively, with $\mathscr{F} \cong \mathscr{S}_{\Sigma,B}(F)$. Let $L \subseteq F^{\mathrm{hom}}$ be a generating set of F, and let $\mathbb{L}$ denote the right filtering ordered set of finite subsets of L. For every $H \in \mathbb{L}$ we denote by F_H the graded sub-$S_{(B)}$-module of F generated by H, which is a graded ideal of $S_{(B)}$ in case b), and by $i_H : \mathscr{S}_{\Sigma,B}(F_H) \rightarrowtail \mathscr{F}$ the induced monomorphism in $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma})$. By [A, II.6.2 Remarque] it holds $F \cong \varinjlim_{H\in\mathbb{L}} F_H$, and thus 3.1.3 implies $\mathscr{F} \cong \mathscr{S}_{\Sigma,B}(F) \cong \varinjlim_{H\in\mathbb{L}} \mathscr{S}_{\Sigma,B}(F_H)$. But by [ÉGA, 0.5.2.3] there exists $H \in \mathbb{L}$ such that i_H is an epimorphism and hence an isomorphism, for $\mathbb{L}$ is right filtering, Y_Σ is quasicompact by I.2.1.2 c), and $\mathscr{F}$ is of finite type. This shows $\mathscr{F} \cong \mathscr{S}_{\Sigma,B}(F_H)$ and thus proves the claim. □

3.4. On injectivity of $\mathscr{S}_\Sigma$

Let $B \subseteq A$ be a big subgroup. If no confusion can arise, then we set $I_B := I_{\Sigma,B}(R)$.

As in the projective case the functors $\mathscr{S}_{\Sigma,B}$, and also their restrictions to graded ideals of S, are not necessarily injective. But we get a positive result if we consider restrictions to saturated graded ideals, where saturation is taken with respect to the irrelevant ideal introduced in 2.1.11.

(3.4.1) We denote by

$$\mathbb{J}^{\mathrm{sat}}_{\Sigma,B}(R) := \{\mathfrak{a} \in \mathbb{J}_{\Sigma,B}(R) \mid \mathfrak{a} = \mathfrak{a}^{\mathrm{sat},I_B}\}$$

the set of I_B-saturated graded ideals of $S_{(B)}$, and if no confusion can arise then we write $\mathbb{J}^{\mathrm{sat}}_{\Sigma,B}$ instead of $\mathbb{J}^{\mathrm{sat}}_{\Sigma,B}(R)$. Since I_B is finitely generated by 2.1.11, it follows from III.3.2.2 that for $\mathfrak{a} \in \mathbb{J}_{\Sigma,B}$ it holds $\mathfrak{a}^{\mathrm{sat},I_N} \in \mathbb{J}^{\mathrm{sat}}_{\Sigma,B}$.

By restriction, the map $\Xi_{\Sigma,B} : \mathbb{J}_{\Sigma,B} \to \widetilde{\mathbb{J}}_\Sigma$ induces a map

$$\Xi^{\mathrm{sat}}_{\Sigma,B,R} : \mathbb{J}^{\mathrm{sat}}_{\Sigma,B}(R) \to \widetilde{\mathbb{J}}_\Sigma(R),$$

and if no confusion can arise then we denote this by $\Xi^{\mathrm{sat}}_{\Sigma,B}$.

In order to get injectivity statements as desired, the subgroup $B \subseteq A$ needs not only be big in A, but also to be contained in a certain subgroup of A.

(3.4.2) Proposition *Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{J}_{\Sigma,B}$.*

a) If $\mathfrak{a}^{\mathrm{sat},I_B} = \mathfrak{b}^{\mathrm{sat},I_B}$, then it holds $\mathscr{S}_{\Sigma,B}(\mathfrak{a}) = \mathscr{S}_{\Sigma,B}(\mathfrak{b})$.

b) Suppose that $B \subseteq \bigcap_{\sigma\in\Sigma}\langle\{\delta_\rho \mid \rho \in \Sigma_1 \setminus \sigma_1\}\rangle_{\mathbb{Z}}$. Then, it holds $\mathfrak{a}^{\mathrm{sat},I_B} = \mathfrak{b}^{\mathrm{sat},I_B}$ if and only if $\mathscr{S}_{\Sigma,B}(\mathfrak{a}) = \mathscr{S}_{\Sigma,B}(\mathfrak{b})$.

Proof. a) Suppose that $\mathfrak{a}^{\mathrm{sat},I_B} = \mathfrak{b}^{\mathrm{sat},I_B}$. Let $\sigma \in \Sigma$, and let $u \in \mathfrak{a}_{(\sigma)}$. Then, there are $k \in \mathbb{N}_0$ with $k\widehat{\alpha}_\sigma \in B$ and $f \in \mathfrak{a}_{k\widehat{\alpha}_\sigma}$ such that $u = \frac{f}{\widehat{Z}^k_\sigma}$, and it follows $f \in \mathfrak{a}_{k\widehat{\alpha}_\sigma} \subseteq (\mathfrak{a}^{\mathrm{sat},I_B})_{k\widehat{\alpha}_\sigma} = (\mathfrak{b}^{\mathrm{sat},I_B})_{k\widehat{\alpha}_\sigma}$. Therefore, there is an $l \in \mathbb{N}_0$ with $l\widehat{\alpha}_\sigma \in B$ and $\widehat{Z}^l_\sigma f \in \mathfrak{b}$, and this yields $u = \frac{f}{\widehat{Z}^k_\sigma} = \frac{\widehat{Z}^l_\sigma f}{\widehat{Z}^{k+l}_\sigma} \in \mathfrak{b}_{(\sigma)}$. So, we have shown that $\mathfrak{a}_{(\sigma)} \subseteq \mathfrak{b}_{(\sigma)}$, and by reasons of symmetry this implies $\mathfrak{a}_{(\sigma)} = \mathfrak{b}_{(\sigma)}$. Now, we get $\mathscr{S}_{\Sigma,B}(\mathfrak{a}) = \mathscr{S}_{\Sigma,B}(\mathfrak{b})$ immediately from the definition of $\mathscr{S}_{\Sigma,B}$.

b) Suppose that $\mathscr{S}_{\Sigma,B}(\mathfrak{a}) = \mathscr{S}_{\Sigma,B}(\mathfrak{b})$. Then, it holds $\mathfrak{a}_{(\sigma)} = \mathfrak{b}_{(\sigma)}$. Let $\alpha \in B$, and let $f \in \mathfrak{a}_\alpha$. There exists $k \in \mathbb{N}$ such that for every $\sigma \in \Sigma$ it holds $k\widehat{\alpha}_\sigma \in B$, and hence it holds $k\widehat{\alpha}_\sigma + \alpha \in B$. By hypothesis there exists for every $\sigma \in \Sigma$ a family $(r_\rho)_{\rho\in\Sigma_1\setminus\sigma_1}$ in $\mathbb{Z}$ such that $k\widehat{\alpha}_\sigma + \alpha = \sum_{\rho\in\Sigma_1\setminus\sigma_1} r_\rho\delta_\rho$. Therefore, for every $\sigma \in \Sigma$ we have $g_\sigma := \frac{\widehat{Z}^k_\sigma \cdot f}{\prod_{\rho\in\Sigma_1\setminus\sigma_1} Z^{r_\rho}_\rho} \in \mathfrak{a}_{(\sigma)} = \mathfrak{b}_{(\sigma)}$, and hence there exist $l \in \mathbb{N}$ with $l\widehat{\alpha}_\sigma \in B$ and $h_\sigma \in \mathfrak{b}_{l\widehat{\alpha}_\sigma}$ such that $g_\sigma = \frac{h_\sigma}{\widehat{Z}^l_\sigma}$. Thus, for every $\sigma \in \Sigma$ it holds $\widehat{Z}^{l+k}_\sigma \cdot f = (\prod_{\rho\in\Sigma_1\setminus\sigma_1} Z^{r_\rho}_\rho)h_\sigma \in \mathfrak{b}$, and therefore there exists $m \in \mathbb{N}$ with $I^m_B \cdot f \subseteq \mathfrak{b}$, hence $f \in \mathfrak{b}^{\mathrm{sat},I_B}$. From this we easily get $\mathfrak{a}^{\mathrm{sat},I_B} \subseteq \mathfrak{b}^{\mathrm{sat},I_B}$ on use of 2.1.11 and III.3.2.2, and then the claim follows by reasons of symmetry and a). □

(3.4.3) Theorem *a) The map $\Xi^{\mathrm{sat}}_{\Sigma,B} : \mathbb{J}^{\mathrm{sat}}_{\Sigma,B} \to \widetilde{\mathbb{J}}_\Sigma$ is surjective.*

b) If $B \subseteq \bigcap_{\sigma\in\Sigma}\langle\{\delta_\rho \mid \rho \in \Sigma_1 \setminus \sigma_1\}\rangle_{\mathbb{Z}}$, then the map $\Xi^{\mathrm{sat}}_{\Sigma,B} : \mathbb{J}^{\mathrm{sat}}_{\Sigma,B} \to \widetilde{\mathbb{J}}_\Sigma$ is bijective.

Proof. a) Let $\mathscr{J} \in \widetilde{\mathbb{J}}_\Sigma$. By 3.3.7 there exists $\mathfrak{a} \in \mathbb{J}_{\Sigma,B}$ with $\mathscr{J} = \mathscr{S}_\Sigma(\mathfrak{a})$, and then 3.4.1 implies $\mathfrak{a}^{\mathrm{sat},I_B} \in \mathbb{J}^{\mathrm{sat}}_{\Sigma,B}$. Moreover, by 3.4.2 a) we get

$$\mathscr{S}_{\Sigma,B}(\mathfrak{a}^{\mathrm{sat},I_B}) = \mathscr{S}_{\Sigma,B}(\mathfrak{a}) = \mathscr{J},$$

and thus $\Xi^{\mathrm{sat}}_{\Sigma,B}$ is surjective.

b) For $\mathfrak{a}, \mathfrak{b} \in \mathbb{J}^{\mathrm{sat}}_{\Sigma,B}$ with $\mathscr{S}_{\Sigma,B}(\mathfrak{a}) = \mathscr{S}_{\Sigma,B}(\mathfrak{b})$ we get $\mathfrak{a} = \mathfrak{a}^{\mathrm{sat},I_B} = \mathfrak{b}^{\mathrm{sat},I_B} = \mathfrak{b}$ by 3.4.2 b), and hence $\Xi^{\mathrm{sat}}_{\Sigma,B}$ is injective. □

Now, we show that the Picard group P of a simplicial fan Σ fulfils all the conditions on B in the above results, and hence we can apply 3.4.3 to get the desired generalisation of the results of Cox and Mustaţă.

(3.4.4) Lemma *Suppose that Σ is full. Let $\sigma \in \Sigma$. Then, it holds $P \subseteq \langle\{\delta_\rho \mid \rho \in \Sigma_1 \setminus \sigma_1\}\rangle_{\mathbb{Z}}$.*

Proof. Let $\alpha \in P$. From 3.1.11 we get the existence of a virtual polytope $p = (m_\tau + \tau^\vee)_{\tau\in\Sigma}$ over Σ such that $m_\tau = 0$ for every $\tau \in \mathrm{face}(\sigma)$ and that $e(p) = \alpha$, and it holds $d(p) = (\rho_N(m_\rho))_{\rho\in\Sigma_1} \in \mathbb{Z}^{\Sigma_1} \cap c^{-1}(\alpha)$ with $\rho_N(m_\rho) = 0$ for every $\rho \in \Sigma_1 \setminus \sigma_1$. Therefore, we get $\alpha = c((\rho_N(m_\rho))_{\rho\in\Sigma_1}) = \sum_{\rho\in\Sigma_1\setminus\sigma_1} \rho_N(m_\rho)\delta_\rho$ as desired. □

(3.4.5) Corollary *Suppose that Σ is full and simplicial.*

a) For $\mathfrak{a}, \mathfrak{b} \in \mathbb{J}_{\Sigma,P}$ it holds $\mathfrak{a}^{\mathrm{sat},I_P} = \mathfrak{b}^{\mathrm{sat},I_P}$ if and only if $\mathscr{S}_{\Sigma,P}(\mathfrak{a}) = \mathscr{S}_{\Sigma,P}(\mathfrak{b})$.

b) The map $\Xi^{\mathrm{sat}}_{\Sigma,P} : \mathbb{J}^{\mathrm{sat}}_{\Sigma,P} \to \widetilde{\mathbb{J}}_\Sigma$ is bijective.

Proof. Clear from 2.2.4, 3.4.4, 3.4.2 b) and 3.4.3 b). □

4. Cohomology of toric schemes

Let V be an $\mathbb{R}$-vector space of finite dimension, let $n := \dim_{\mathbb{R}}(V)$, let N be a $\mathbb{Z}$-structure on V, let $M := N^*$, and let Σ be an N-fan in V. If no confusion can arise, then we set $A := A_{\Sigma_1}$.
Furthermore, let R be a ring. If no confusion can arise, then we set $S := S_{\Sigma_1}(R)$ and $Y_\Sigma := Y_\Sigma(R)$, and $Y_\sigma := Y_\sigma(R)$ for every $\sigma \in \Sigma$.

4.1. The second total functor of sections

Let $B \subseteq A$ be a big subgroup. If no confusion can arise, then we set $S_{B,\sigma} := (S_{(B)})_\sigma$ for every $\sigma \in \Sigma$.

As mentioned in 3.2, we introduce now the *second* total functor of sections. In contrast to the first one (and also to its name), this functor is defined on $\mathsf{GrMod}^B(S_{(B)})$ instead on $\mathsf{QCMod}(\mathscr{O}_{Y_\Sigma})$. However, by 3.3.6 this difference is rather a technical one.

(4.1.1) If $\alpha \in B$, then by III.1.2.4, III.2.3.11 and 3.1.3 we have the exact functor

$$\mathscr{S}_{\Sigma,B}(\bullet(\alpha)) = \mathscr{S}_{\Sigma,B}(S_{(B)}(\alpha) \otimes_{S_{(B)}} \bullet) : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}).$$

If moreover $U \subseteq Y_\Sigma$ is an open subset, then composition with the left exact functor

$$\Gamma(U, \bullet) : \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}) \to \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}(U))$$

yields a left exact functor

$$\Gamma_\alpha^{\Sigma,R,B,U}(\bullet) := \Gamma(U, \mathscr{S}_{\Sigma,B}(\bullet(\alpha))) : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}(U)).$$

If $V \subseteq Y_\Sigma$ is a further open subset with $V \subseteq U$, then restriction from U to V induces a morphism

$$\Gamma_\alpha^{\Sigma,R,B,U} \to \Gamma_\alpha^{\Sigma,R,B,V}$$

in $\mathsf{Hom}(\mathsf{GrMod}^B(S_{(B)}), \mathsf{Mod}(\mathscr{O}_{Y_\Sigma(U)}))$, denoted by $\upharpoonright_V$ if no confusion can arise.

Taking direct sums over all $\alpha \in B$ yields for every open subset $U \subseteq Y_\Sigma$ a left exact functor

$$\Gamma_{**}^{\Sigma,R,B,U} := \bigoplus_{\alpha \in B} \Gamma_\alpha^{\Sigma,R,B,U} : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}(U)),$$

and for every further open subset $V \subseteq Y_\Sigma$ with $V \subseteq U$ a morphism

$$\Gamma_{**}^{\Sigma,R,B,U} \to \Gamma_{**}^{\Sigma,R,B,V},$$

denoted again by $\upharpoonright_V$ if no confusion can arise. The functor $\Gamma_{**}^{\Sigma,R,B,U}$ is called *the second total functor of sections over U associated with Σ and B over R.*

If no confusion can arise, then we set $\Gamma_{**}^{\Sigma,B,U} := \Gamma_{**}^{\Sigma,R,B,U}$, $\Gamma_{**}^{\Sigma,B} := \Gamma_{**}^{\Sigma,B,Y_\Sigma}$ and $\Gamma_{**}^{\Sigma} := \Gamma_{**}^{\Sigma,A}$.

(4.1.2) If $U \subseteq Y_\Sigma$ is an open subset, then it follows from III.1.4.6, 3.1.5 and 3.1.6 that

$$\Gamma_{**}^{\Sigma,B,U}(S_{(B)}) = \bigoplus_{\alpha\in B} \Gamma(U, \mathscr{S}_{\Sigma,B}(S_{(B)}(\alpha))) = \bigoplus_{\alpha\in B} \Gamma(U, \mathscr{S}_{\Sigma,B}(S(\alpha)_{(B)})) =$$

$$\bigoplus_{\alpha\in B} \Gamma(U, \mathscr{S}_{\Sigma}(S(\alpha))) = \bigoplus_{\alpha\in B} \Gamma_*^{\Sigma,U}(\mathscr{O}_{Y_\Sigma})_\alpha = \Gamma_*^{\Sigma,U}(\mathscr{O}_{Y_\Sigma})_{(B)},$$

and hence $\Gamma_{**}^{\Sigma,B,U}(S_{(B)})$ carries a structure of B-graded ring.

Moreover, if $V \subseteq Y_\Sigma$ is a further open subset with $V \subseteq U$, then the restriction morphism $\Gamma_{**}^{\Sigma,B,U}(S_{(B)}) \to \Gamma_{**}^{\Sigma,B,V}(S_{(B)})$ equals the B-restriction of the restriction morphism $\Gamma_*^{\Sigma,U}(\mathscr{O}_{Y_\Sigma}) \to \Gamma_*^{\Sigma,V}(\mathscr{O}_{Y_\Sigma})$ from 3.2.4, and in particular it is a morphism in GrAnn^B.

(4.1.3) Let F be a B-graded $S_{(B)}$-module, and let $\alpha, \beta \in B$. For every $\sigma \in \Sigma$, the structure of B-graded $S_{B,\sigma}$-module on F_σ yields a morphism

$$(S_{B,\sigma})_\alpha \otimes_{S_{(\sigma)}} (F_\sigma)_\beta \to (F_\sigma)_{\alpha+\beta}$$

in $\mathsf{Mod}(S_{(\sigma)})$ such that for every $\tau \in \mathrm{face}(\sigma)$ the diagram

$$\begin{tikzcd}
(S_{B,\sigma})_\alpha \otimes_{S_{(\sigma)}} (F_\sigma)_\beta \arrow[r] \arrow[d, "(\eta_\tau^\sigma)_\alpha \otimes_{S_{(\sigma)}} (\eta_\tau^\sigma(F))_\beta"'] & (F_\sigma)_{\alpha+\beta} \arrow[d, "(\eta_\tau^\sigma(F))_{\alpha+\beta}"] \\
(S_{B,\tau})_\alpha \otimes_{S_{(\tau)}} (F_\tau)_\beta \arrow[r] & (F_\tau)_{\alpha+\beta}
\end{tikzcd}$$

in $\mathsf{Mod}(S_{(\sigma)})$ commutes. These morphisms induce a morphism

$$\mathscr{S}_{\Sigma,B}(S_{(B)}(\alpha)) \otimes_{\mathscr{O}_{Y_\Sigma}} \mathscr{S}_{\Sigma,B}(F(\beta)) \to \mathscr{S}_{\Sigma,B}(F(\alpha+\beta))$$

in $\mathsf{Mod}(\mathscr{O}_{Y_\Sigma})$.

If $U \subseteq Y_\Sigma$ is an open subset, then composing $\Gamma(U, \bullet)$ with the above morphisms yields a structure of B-graded $\Gamma_{**}^{\Sigma,B,U}(S_{(B)})$-module on $\Gamma_{**}^{\Sigma,B,U}(F)$. By means of this we always consider $\Gamma_{**}^{\Sigma,B,U}$ as a functor from $\mathsf{GrMod}^B(S_{(B)})$ to $\mathsf{GrMod}^B(\Gamma_{**}^{\Sigma,B,U}(S_{(B)}))$, and then it is again left exact. It is readily checked that if $V \subseteq Y_\Sigma$ is a further open subset with $V \subseteq U$, then the restriction morphism $\upharpoonright_V\colon \Gamma_{**}^{\Sigma,B,U} \to \Gamma_{**}^{\Sigma,B,V}$ is a morphism of functors from $\mathsf{GrMod}^B(S_{(B)})$ to $\mathsf{GrMod}^B(\Gamma_{**}^{\Sigma,B,U}(S_{(B)}))$.

(4.1.4) Let $U \subseteq Y_\Sigma$ be an open subset. We consider the B-restriction of the morphism $\eta_\Sigma : S \to \Gamma_*^\Sigma(\mathscr{O}_{Y_\Sigma})$ from 3.2.6. By 4.1.2 this is a morphism $S_{(B)} \to \Gamma_{**}^{\Sigma,B}(S_{(B)})$ in GrAnn^B. Composition with the restriction morphism to U yields a morphism $S_{(B)} \to \Gamma_{**}^{\Sigma,B,U}(S_{(B)})$ in GrAnn^B, by means of which we consider $\Gamma_{**}^{\Sigma,B,U}(S_{(B)})$ as a B-graded $S_{(B)}$-algebra. Moreover, by scalar restriction we consider $\Gamma_{**}^{\Sigma,B,U}$ as a functor from $\mathsf{GrMod}^B(S_{(B)})$ to $\mathsf{GrMod}^B(S_{(B)})$, and then it is again left exact.

Now we establish the existence of a canonical morphism from the first to the second total functor of sections, induced by morphisms met already in Section III.2.5. As the example given below shows, this is not necessarily an isomorphism.

(4.1.5) If $\alpha \in B$, then by III.2.5.8 and III.2.5.9 we have for every $\sigma \in \Sigma$ a morphism

$$\delta_{\sigma,\alpha} := \delta_{\widehat{Z}_\sigma}(S(\alpha), \bullet) : S(\alpha)_{(\sigma)} \otimes_{S_{(\sigma)}} \bullet_{(\sigma)} \to (S(\alpha) \otimes_S \bullet)_{(\sigma)}$$

in $\mathsf{Hom}(\mathsf{GrMod}^B(S_{(B)}, \mathsf{Mod}(S_{(\sigma)}))$ such that for all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ the diagram

$$\begin{tikzcd}
S(\alpha)_{(\sigma)} \otimes_{S_{(\sigma)}} \bullet_{(\sigma)} \arrow[r, "\delta_{\sigma,\alpha}"] \arrow[d, "\eta^{(\sigma)}_{(\tau)} \otimes_{S_{(\sigma)}} \eta^{(\sigma)}_{(\tau)}"'] & (S(\alpha) \otimes_S \bullet)_{(\sigma)} \arrow[d, "\eta^{(\sigma)}_{(\tau)}"] \\
S(\alpha)_{(\tau)} \otimes_{S_{(\tau)}} \bullet_{(\tau)} \arrow[r, "\delta_{\tau,\alpha}"] & (S(\alpha) \otimes_S \bullet)_{(\tau)}
\end{tikzcd}$$

in $\mathsf{Mod}(S_{(\sigma)})$ commutes. These morphisms give rise to a morphism

$$\delta_{\Sigma,B,\alpha} : \mathscr{S}_{\Sigma,B}(\bullet)(\alpha) \to \mathscr{S}_{\Sigma,B}(\bullet(\alpha))$$

in $\mathsf{Hom}(\mathsf{GrMod}^B(S_{(B)}), \mathsf{Mod}(\mathscr{O}_{Y_\Sigma}))$, and by taking sections over an open subset $U \subseteq Y_\Sigma$ and direct sums over all $\alpha \in B$ we get a morphism

$$\delta_{\Sigma,B,U} : \Gamma_*^{\Sigma,U}(\mathscr{S}_{\Sigma,B}(\bullet))_{(B)} \to \Gamma_{**}^{\Sigma,B,U}(\bullet)$$

in $\mathsf{Hom}(\mathsf{GrMod}^B(S_{(B)}), \mathsf{GrMod}^B(\Gamma_{**}^{\Sigma,B,U}(S_{(B)}))$ such that

$$\delta_{\Sigma,B,U}(S) = \mathrm{Id}_{\Gamma_*^{\Sigma,U}(\mathscr{O}_{Y_\Sigma})_{(B)}}$$

(see 4.1.2). Moreover, for every further open subset $V \subseteq Y_\Sigma$ with $V \subseteq U$, the diagram

$$\begin{tikzcd}
\Gamma_*^{\Sigma,U}(\mathscr{S}_{\Sigma,B}(\bullet))_{(B)} \arrow[r, "\delta_{\Sigma,B,U}"] \arrow[d, "(\upharpoonright_V \circ \mathscr{S}_{\Sigma,B})_{(B)}"'] & \Gamma_{**}^{\Sigma,B,U}(\bullet) \arrow[d, "\upharpoonright_V"] \\
\Gamma_*^{\Sigma,V}(\mathscr{S}_{\Sigma,B}(\bullet))_{(B)} \arrow[r, "\delta_{\Sigma,B,V}"] & \Gamma_{**}^{\Sigma,B,V}(\bullet)
\end{tikzcd}$$

in $\mathsf{Hom}(\mathsf{GrMod}^B(S_{(B)}), \mathsf{GrMod}^B(\Gamma_{**}^{\Sigma,B,U}(S_{(B)})))$ commutes.

(4.1.6) The morphism

$$\delta_{\Sigma,B,U} : \Gamma_*^{\Sigma,U}(\mathscr{S}_{\Sigma,B}(\bullet))_{(B)} \to \Gamma_{**}^{\Sigma,B,U}(\bullet)$$

of functors from 4.1.5 is not necessarily an isomorphism. We give a counterexample on the Cox scheme over $\mathbb{Z}$ associated with the complete $\mathbb{Z}^2$-fan Σ in $\mathbb{R}^2$ defined in II.4.2.2 with $k = 3$. Its Cox ring is $S = \mathbb{Z}[Z_1, Z_2, Z_3, Z_4]$ with $\deg(Z_1) = (0, 1)$, $\deg(Z_2) = (1, -3)$, $\deg(Z_3) = (0, 1)$ and $\deg(Z_4) = (1, 0)$, where $A = \mathbb{Z}^2$ (see 2.1.5).

Now, we consider the graded ideal $\mathfrak{a} := \langle Z_1^4 Z_2 Z_3 Z_4 \rangle_S \subseteq S$, and we show that $\delta_{\Sigma,A,Y_\Sigma}(\mathfrak{a})$ is not an isomorphism. In order to do this it suffices to show that the component of some degree $\alpha \in A$ of the restriction to $Y_\sigma(\mathbb{Z})$ for some $\sigma \in \Sigma$ of $\delta_{\Sigma,A,Y_\Sigma}(\mathfrak{a})$, or – equivalently –, the morphism

$$\delta_{\sigma,\alpha}(\mathfrak{a}) : S(\alpha)_{(\sigma)} \otimes_{S_{(\sigma)}} \mathfrak{a}_{(\sigma)} \to \mathfrak{a}(\alpha)_{(\sigma)}$$

induced by the structure of S_σ-module on $\mathfrak{a}_\sigma$, is not an isomorphism. We will do this for $\alpha := (3,5)$ and $\sigma := \mathrm{cone}((-1,3),(0,-1))$. It holds $\widehat{Z}_\sigma = Z_1Z_2$ and hence $\deg(\widehat{Z}_\sigma) = (1,-2)$, and we will show that $\mathrm{Im}(\delta_{\sigma,\alpha}(\mathfrak{a}))$ does not contain $f := \frac{(Z_1^2Z_2Z_3^2Z_4)\otimes(Z_1^4Z_2Z_3Z_4)}{Z_1Z_2} \in (S(\alpha)\otimes_S \mathfrak{a})_{(\sigma)}$.

So, we assume that $f \in \mathrm{Im}(\delta_{\sigma,\alpha}(\mathfrak{a}))$. Then, there are $l,k \in \mathbb{N}_0$, families $(r_j)_{j=1}^l$ and $(s_j)_{j=1}^l$ in $\mathbb{Z}$, a family $(u_j)_{j=1}^l$ of monomials in $S_{k\deg(Z_1Z_2)}$ and a family $(t_j)_{j=1}^l$ of monomials in $\mathfrak{a}_{k\deg(Z_1Z_2)}$ with $\frac{s_ju_j}{(Z_1Z_2)^k} \in S(\alpha)_{(\sigma)}$ and $\frac{r_jt_j}{(Z_1Z_2)^k} \in \mathfrak{a}_{(\sigma)}$ for every $j \in [1,l]$ such that $f = \sum_{j=1}^l \frac{s_ju_j\otimes r_jt_j}{(Z_1Z_2)^{2k}}$. Since $f \neq 0$ and by identification of $S(\alpha)\otimes_S \mathfrak{a}$ (see III.2.3.11) and $\mathfrak{a}(\alpha)$ it follows that there is an $i \in [1,l]$ with $Z_1^{6+2k}Z_2^{2+2k}Z_3^3Z_4^2 = Z_1Z_2t_iu_i$. Furthermore, there are families $(a_j)_{j=1}^4$ and $(b_j)_{j=1}^4$ in $\mathbb{N}_0$ with $t_j = \prod_{j=1}^4 Z_j^{a_j}$ and $u_j = \prod_{j=1}^4 Z_j^{b_j}$. From $\deg(t_i) = k\deg(Z_1Z_2)$, $\deg(u_i) - (3,5) = k\deg(Z_1Z_2)$ and $t_i \in \mathfrak{a}$ we get the equations

$$a_1+b_1 = 2k+5,\ a_2+b_2 = 2k+1,\ a_3+b_3 = 3,\ a_4+b+4 = 2,\ a_2+a_4 = k,$$

$$b_2+b_4 = k+3,\ a_1+a_3-3a_2 = -2k,\ b_1+b_3-3b_2 = -2k+5$$

with $b_1,b_2,b_3,b_4,k \in \mathbb{N}_0$, $a_1 \in \mathbb{N}_{\geq 4}$ and $a_2,a_3,a_4 \in \mathbb{N}$. These imply the equations $a_4 = k-2$ and $a_3 = 3a_2 - a_1 - 2k$, and the inequalities

$$2k+5-a_1 \geq 0,\ 2k+1-a_2 \geq 0,\ 3-a_3 \geq 0,\ 2-a_4 \geq 0$$

with $k \in \mathbb{N}_0$, $a_1 \in \mathbb{N}_{\geq 4}$ and $a_2,a_3,a_4 \in \mathbb{N}$. Therefore, we get the inequalities

$$3a_2 - a_1 \geq 2k+1,\ k-1 \geq a_2,\ 2k+5 \geq a_1,$$

$$2k+1 \geq a_2,\ 3-2k \geq 3a_2-a_1,\ 2-k \geq a_2$$

with $k \in \mathbb{N}_0$, $a_1 \in \mathbb{N}_{\geq 4}$ and $a_2 \in \mathbb{N}$, in particular $3-2k \geq 3a_2-a_1 \geq 2k+1$, hence $2 \geq 4k$ and therefore $k=0$. But this implies the contradiction $-1 = k-1 \geq a_2$, and thus our claim is proven.

4.2. The toric Serre-Grothendieck correspondence

Let $B \subseteq A$ be a big subgroup. If no confusion can arise, then we set $I_B := I_{\Sigma,B}(R)$ and $I := I_A$.

Sheaf cohomology is defined by taking the right derived cohomological functor of the functor of sections. So, if we take the right derived cohomological functor of the second total functor of sections we get functors that contain sheaf cohomology as a graded component.

(4.2.1) Let $U \subseteq Y_\Sigma$ be an open subset, and let $\alpha \in B$. The category $\mathsf{GrMod}^B(S_{(B)})$ is Abelian and fulfils AB5 by III.2.1.1 a), and the functor

$$\Gamma_\alpha^{\Sigma,B,U} : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{GrMod}^B(S_0)$$

is left exact (see 4.1.1). Hence, we can consider its right derived cohomological functor, denoted by $(H^i_{\alpha,\Sigma,B,U})_{i\in\mathbb{Z}}$. If F is a B-graded $S_{(B)}$-module and $i \in \mathbb{Z}$, then it follows from 3.1.3 and [6, 2.2.1] that

$$H^i_{\alpha,\Sigma,B,U}(F) = H^i(U, \mathscr{S}_{\Sigma,B}(F(\alpha)))$$

is the i-th sheaf cohomology of $\mathscr{S}_\Sigma(F(\alpha))$ over U in the sense of [ÉGA, 0.12.1]. In particular,

$$H^i_{0,\Sigma,B,U}(F) = H^i(U, \mathscr{S}_{\Sigma,B}(F))$$

is the i-th sheaf cohomology of $\mathscr{S}_{\Sigma,B}(F)$ over U. On use of [6, 2.3] we can and do identify $H^0_{\alpha,\Sigma,B,U}$ with $\Gamma^{\Sigma,B,U}_\alpha$.

If no confusion can arise, then we set $H^i_{\alpha,\Sigma,B} := H^i_{\alpha,\Sigma,B,Y_\Sigma}$, $H^i_{\alpha,\Sigma,U} := H^i_{\alpha,\Sigma,A,U}$ and $H^i_{\alpha,\Sigma} := H^i_{\alpha,\Sigma,A}$ for every $i \in \mathbb{Z}$.

(4.2.2) Let $U \subseteq Y_\Sigma$ be an open subset. The category $\mathsf{GrMod}^B(S_{(B)})$ is Abelian and fulfils AB5 by III.2.1.1 a), and the functor

$$\Gamma^{\Sigma,B,U}_{**} : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{GrMod}^B(S_{(B)})$$

is left exact (see 4.1.1). Hence, we can consider its right derived cohomological functor, denoted by $(H^i_{**,\Sigma,B,U})_{i\in\mathbb{Z}}$. Since $\mathsf{GrMod}^B(S_{(B)})$ fulfils AB4 it is seen on use of [6, 2.2.1] that

$$H^i_{**,\Sigma,B,U} = \bigoplus_{\alpha\in B} H^i_{\alpha,\Sigma,B,U}$$

for every $i \in \mathbb{Z}$. On use of [6, 2.3] we can and do identify $H^0_{**,\Sigma,B,U}$ with $\Gamma^{\Sigma,B,U}_{**}$.

If no confusion can arise, then we set $H^i_{**,\Sigma,B} := H^i_{**,\Sigma,B,Y_\Sigma}$, $H^i_{**,\Sigma,U} := H^i_{**,\Sigma,A,U}$ and $H^i_{**,\Sigma} := H^i_{**,\Sigma,A}$ for every $i \in \mathbb{Z}$.

Next, we show that the cohomology functors $H^i_{**,\Sigma,U}$ commute with restriction of degrees to the big subgroup $B \subseteq A$.

(4.2.3) Proposition *For every open subset $U \subseteq Y_\Sigma$ and for every $i \in \mathbb{Z}$ it holds*

$$H^i_{**,\Sigma,U}(\bullet)_{(B)} = H^i_{**,\Sigma,B,U}(\bullet_{(B)}).$$

Proof. For every $\alpha \in B$ it holds

$$H^i_{\alpha,\Sigma,U}(\bullet) = H^i(U, \mathscr{S}_\Sigma(\bullet(\alpha))) = H^i(U, \mathscr{S}_{\Sigma,B}(\bullet(\alpha)_{(B)})) =$$

$$H^i(U, \mathscr{S}_{\Sigma,B}(\bullet_{(B)}(\alpha))) = H^i_{\alpha,\Sigma,B,U}(\bullet_{(B)})$$

by 4.2.1, 3.1.5 and III.1.4.6, and the claim follows from this. □

We aim at the *toric Serre-Grothendieck correspondence,* giving a relation between sheaf cohomology (that is, the functors $H^i_{**,\Sigma,B}$ as introduced above) and graded local cohomology with respect to the irrelevant ideal, analogously to the case of projective schemes and positively graded rings as treated in [2, 20.4.4]. We follow the proof given there. Hence, we will use the characterisation of ideal transformation functors from III.4.4.17, leading to an ITR-hypothesis.

(4.2.4) Let F be a B-graded $S_{(B)}$-module. For every $\alpha \in B$ and all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$, we have the commutative diagram

$$\begin{tikzcd} F_\alpha \arrow[r, "\eta_\sigma(F)_\alpha"] \arrow[dr, "\eta_\tau(F)_\alpha"'] & (F_\sigma)_\alpha \arrow[d, "\eta_\tau^\sigma(F)_\alpha"] \\ & (F_\tau)_\alpha \end{tikzcd}$$

in $\mathsf{Mod}(S_0)$. Since $(F_\sigma)_\alpha = F(\alpha)_\sigma = \Gamma_\alpha^{\Sigma,B,Y_\sigma}(F)$ for every $\sigma \in \Sigma$ there is a unique morphism

$$\overline{\eta}_{\Sigma,\alpha}(F) : F_\alpha \to \Gamma_\alpha^{\Sigma,B}(F)$$

in $\mathsf{Mod}(S_0)$ such that for every $\sigma \in \Sigma$ the diagram

$$\begin{tikzcd} F_\alpha \arrow[r, "\overline{\eta}_{\Sigma,\alpha}(F)"] \arrow[dr, "\eta_\sigma(F)_\alpha"'] & \Gamma_\alpha^{\Sigma,B}(F) \arrow[d, "\upharpoonright_{Y_\sigma}"] \\ & \Gamma_\alpha^{\Sigma,B,Y_\sigma}(F) \end{tikzcd}$$

in $\mathsf{Mod}(S_0)$ commutes. It is easy to see that by taking direct sums over all $\alpha \in B$ and varying F we get a morphism

$$\overline{\eta}_{\Sigma,B} := \bigoplus_{\alpha \in B} \overline{\eta}_{\Sigma,\alpha} : \mathrm{Id}_{\mathsf{GrMod}^B(S_{(B)})} \to \Gamma_{**}^{\Sigma,B}$$

in $\mathsf{Hom}(\mathsf{GrMod}^B(S_{(B)}), \mathsf{GrMod}^B(S_{(B)}))$. If no confusion can arise, then we set $\overline{\eta}_\Sigma := \overline{\eta}_{\Sigma,A}$.

Clearly, $\overline{\eta}_{\Sigma,B}(F)$ equals the B-restriction of

$$\overline{\eta}_\Sigma(F^{(A)}) : F^{(A)} \to \Gamma_{**}^\Sigma(F^{(A)}).$$

(4.2.5) Lemma *It holds*

$${}^B\Gamma_{I_B} \circ \mathrm{Ker}(\overline{\eta}_{\Sigma,B}) = \mathrm{Ker}(\overline{\eta}_{\Sigma,B})$$

and

$${}^B\Gamma_{I_B} \circ \mathrm{Coker}(\overline{\eta}_{\Sigma,B}) = \mathrm{Coker}(\overline{\eta}_{\Sigma,B}).$$

Proof. Let F be a B-graded $S_{(B)}$-module. We have to show that ${}^B\Gamma_{I_B}(\mathrm{Ker}(\overline{\eta}_{\Sigma,B}(F))) = \mathrm{Ker}(\overline{\eta}_{\Sigma,B}(F))$ and that ${}^B\Gamma_{I_B}(\mathrm{Coker}(\overline{\eta}_{\Sigma,B}(F))) = \mathrm{Coker}(\overline{\eta}_{\Sigma,B}(F))$. By 4.2.4 this is equivalent to

$${}^A\Gamma_I(\mathrm{Ker}(\overline{\eta}_\Sigma(F^{(A)}))) = \mathrm{Ker}(\overline{\eta}_\Sigma(F^{(A)}))$$

and

$${}^A\Gamma_I(\mathrm{Coker}(\overline{\eta}_\Sigma(F^{(A)}))) = \mathrm{Coker}(\overline{\eta}_\Sigma(F^{(A)})),$$

and hence we can assume without loss of generality that $B = A$.

First, let $x \in \mathrm{Ker}(\overline{\eta}_{\Sigma}(F))$. We have to show that there exists $m \in \mathbb{N}_0$ such that $I^m x = 0$. For this we can assume without loss of generality that there is an $\alpha \in A$ with $x \in F_{\alpha}$. As $\overline{\eta}_{\Sigma}(F)(x) = 0$ we have

$$0 = \overline{\eta}_{\Sigma}(F)(x) \upharpoonright_{Y_\sigma} = \frac{x}{1} \in (F_\sigma)_\alpha$$

for every $\sigma \in \Sigma$ by definition of $\overline{\eta}_{\Sigma}$, and hence there exists for every $\sigma \in \Sigma$ an $m_\sigma \in \mathbb{N}_0$ such that $\widehat{Z}_\sigma^{m_\sigma} x = 0$. The first claim follows from this.

Next, let $x \in \Gamma_{**}^{\Sigma}(F)$. We have to show that there exists $m \in \mathbb{N}_0$ such that $I^m x \subseteq \mathrm{Im}(\overline{\eta}_{\Sigma}(F))$. For this we can assume without loss of generality that there is an $\alpha \in A$ with $x \in \Gamma_{**}^{\Sigma}(F)_\alpha = \Gamma(Y_\Sigma, \mathscr{S}_\Sigma(F(\alpha)))$. Then, there exist $l \in \mathbb{N}_0$ and for every $\sigma \in \Sigma$ an $x_\sigma \in F_{\alpha + l\widehat{\alpha}_\sigma}$ such that $x \upharpoonright_{Y_\sigma} = \frac{x_\sigma}{\widehat{Z}_\sigma^l} \in (F_\sigma)_\alpha$. Therefore, for every $\sigma \in \Sigma$ it holds

$$(\widehat{Z}_\sigma^l x) \upharpoonright_{Y_\sigma} = \tfrac{x_\sigma}{1} = \overline{\eta}_{\Sigma}(F)(x) \upharpoonright_{Y_\sigma} \in (F_\sigma)_\alpha$$

and hence

$$(\widehat{Z}_\sigma^l x - \overline{\eta}_{\Sigma}(F)(x)) \upharpoonright_{Y_\sigma} = 0 \in (F_\sigma)_\alpha.$$

So, for every $\sigma \in \Sigma$ there exists $m_\sigma \in \mathbb{N}_0$ with $\widehat{Z}_\sigma^{m_\sigma}(\widehat{Z}_\sigma^l x - \overline{\eta}_{\Sigma}(F)(x)) = 0$. Thus, if $m \in \mathbb{N}_0$ fulfils $m \geq m_\sigma$ for every $\sigma \in \Sigma$, then it holds

$$\widehat{Z}_\sigma^m x = \widehat{Z}_\sigma^{m-l-m_\sigma} \widehat{Z}_\sigma^{m_\sigma} \widehat{Z}_\sigma^l x = \widehat{Z}_\sigma^{m-l} \overline{\eta}_{\Sigma}(F)(x) = \overline{\eta}_{\Sigma}(F)(\widehat{Z}_\sigma^m x)$$

for every $\sigma \in \Sigma$, and hence $I^m x \subseteq \mathrm{Im}(\overline{\eta}_{\Sigma}(F))$. The second claim follows from this. □

(4.2.6) Lemma *It holds* ${}^B\Gamma_{I_B} \circ \Gamma_{**}^{\Sigma,B} = 0$.

Proof. Let F be a B-graded $S_{(B)}$-module. We have to show that ${}^B\Gamma_{I_B}(\Gamma_{**}^{\Sigma,B}(F)) = 0$. As this is equivalent to ${}^A\Gamma_I(\Gamma_{**}^{\Sigma}(F^{(A)})) = 0$ we can assume without loss of generality that $B = A$. So, let $x \in \Gamma_{**}^{\Sigma}(F)$ be such that there exists $m \in \mathbb{N}_0$ with $I^m x = 0$. We have to show that $x = 0$, and hence we can assume without loss of generality that there is an $\alpha \in A$ with $x \in \Gamma_{**}^{\Sigma}(F)_\alpha = \Gamma(Y_\Sigma, \mathscr{S}_\Sigma(F(\alpha)))$. Then, there exist $l \in \mathbb{N}_0$ and for every $\sigma \in \Sigma$ an $x_\sigma \in F_{\alpha + l\widehat{\alpha}_\sigma}$ such that $x \upharpoonright_{Y_\sigma} = \frac{x_\sigma}{\widehat{Z}_\sigma^l} \in (F_\sigma)_\alpha$. From this we get

$$x \upharpoonright_{Y_\sigma} = \tfrac{\widehat{Z}_\sigma^m x_\sigma}{\widehat{Z}_\sigma^{m+l}} = \tfrac{1}{\widehat{Z}_\sigma^m}(\widehat{Z}_\sigma^m x) \upharpoonright_{Y_\sigma} = 0$$

for every $\sigma \in \Sigma$ and thus $x = 0$ as desired. □

(4.2.7) Keep in mind that in III.4.4.1 we have defined the ideal transformation functor

$${}^B D_{I_B} : \mathsf{GrMod}^B(S_{(B)}) \to \mathsf{GrMod}^G(S_{(B)})$$

with respect to I_B (see also III.4.4.3), and that in III.4.4.7 we have shown the existence of a canonical exact sequence

$$\mathbb{Y}_{I_B} : 0 \longrightarrow {}^B\Gamma_{I_B} \xrightarrow{\xi_{I_B}} \mathrm{Id}_{\mathsf{GrMod}^B(S_{(B)})} \xrightarrow{\eta_{I_B}} {}^B D_{I_B} \xrightarrow{\zeta_{I_B}} {}^B H^1_{I_B} \longrightarrow 0$$

of functors (see also III.4.4.8).

(4.2.8) Proposition *Suppose that $S_{(B)}$ has the ITR-property with respect to I_B.*

a) There exists a unique morphism $\overline{\eta}'_{\Sigma,B} : \Gamma^{\Sigma,B}_{**} \to {}^BD_{I_B}$ of functors such that the diagram

$$\begin{tikzcd}
\mathrm{Id}_{\mathsf{GrMod}^B(S_{(B)})} \arrow[r, "\overline{\eta}_{\Sigma,B}"] \arrow[dr, "\eta_{I_B}"'] & \Gamma^{\Sigma,B}_{**} \arrow[d, "\overline{\eta}'_{\Sigma,B}"] \\
 & {}^BD_{I_B}
\end{tikzcd}$$

of functors commutes.

b) $\overline{\eta}'_{\Sigma,B}$ is an isomorphism.

Proof. Claim a) follows immediately from 4.2.5 and III.4.4.17 b). Furthermore, 4.2.6 and III.4.4.17 c) imply that $\overline{\eta}'_{\Sigma,B}$ is a monomorphism, and so it remains to show that $\overline{\eta}'_{\Sigma,B}$ is an epimorphism. For this we can assume without loss of generality that $B = A$.

Let F be an A-graded S-module. We set $\overline{\eta} := \overline{\eta}_\Sigma(F)$, $\overline{\eta}' := \overline{\eta}'_{\Sigma,A}(F)$ and $\eta := \eta_I(F)$. It suffices to show that the morphism

$$\overline{\eta}'_\alpha : \Gamma(Y_\Sigma, \mathscr{S}_\Sigma(F(\alpha))) \to (\varinjlim_{m\in\mathbb{N}_0} {}^A\mathrm{Hom}_S(I^m, F))_\alpha$$

in $\mathsf{Mod}(S_0)$ is surjective for every $\alpha \in A$. So, let $\alpha \in A$. By III.2.1.1 c) we have

$$(\varinjlim_{m\in\mathbb{N}_0} {}^A\mathrm{Hom}_S(I^m, F))_\alpha = \varinjlim_{m\in\mathbb{N}_0} ({}^A\mathrm{Hom}_S(I^m, F)_\alpha) =$$

$$\varinjlim_{m\in\mathbb{N}_0} \mathrm{Hom}_{\mathsf{GrMod}^A(S)}(I^m, F(\alpha)) = \varinjlim_{m\in\mathbb{N}_0} ({}^A\mathrm{Hom}_S(I^m, F(\alpha))_0),$$

and hence we can assume without loss of generality that $\alpha = 0$, that is, we have to show that the morphism

$$\overline{\eta}'_0 : \Gamma(Y_\Sigma, \mathscr{S}_\Sigma(F)) \to \varinjlim_{m\in\mathbb{N}_0} \mathrm{Hom}_{\mathsf{GrMod}^A(S)}(I^m, F)$$

in $\mathsf{Mod}(S_0)$ is surjective.

For every $l \in \mathbb{N}_0$ we denote by

$$\iota_l : \mathrm{Hom}_{\mathsf{GrMod}^A(S)}(I^l, F) \to \varinjlim_{m\in\mathbb{N}_0} \mathrm{Hom}_{\mathsf{GrMod}^A(S)}(I^m, F)$$

the canonical morphism in $\mathsf{Mod}(S_0)$. So, for every $x \in F_0$ the morphism $I^l \to F$, $a \mapsto ax$ in $\mathsf{GrMod}^A(S)$ is mapped by ι_l onto $\eta(x)$.

Now, let $x \in \varinjlim_{m\in\mathbb{N}_0} \mathrm{Hom}_{\mathsf{GrMod}^A(S)}(I^m, F)$. We construct a preimage of y under $\overline{\eta}'$. First, there exist $l \in \mathbb{N}_0$ and $h \in \mathrm{Hom}_{\mathsf{GrMod}^A(S)}(I^l, F)$ such that $x = \iota_l(h)$. For every $\sigma \in \Sigma$ we set $y_\sigma := \frac{h(\widehat{Z}^l_\sigma)}{\widehat{Z}^l_\sigma} \in F_{(\sigma)} = \Gamma(Y_\sigma, \mathscr{S}_\Sigma(F))$. For all $\sigma, \tau \in \Sigma$ with $\tau \preccurlyeq \sigma$ it holds

$$y_\sigma \upharpoonright_{Y_\tau} = \frac{h(\widehat{Z}^l_\sigma)\widehat{Z}^l_\tau}{\widehat{Z}^l_\sigma \widehat{Z}^l_\tau} = \frac{h(\widehat{Z}^l_\tau)\widehat{Z}^l_\sigma}{\widehat{Z}^l_\tau \widehat{Z}^l_\sigma} = y_\tau.$$

Therefore, there exists a unique $y \in \Gamma(Y_\Sigma, \mathscr{S}_\Sigma(F))$ such that $y \upharpoonright_{Y_\sigma} = y_\sigma$ for every $\sigma \in \Sigma$.

Next, let $r \in I^m$. For every $\sigma \in \Sigma$ it holds

$$(ry) \upharpoonright_{Y_\sigma} = r(y \upharpoonright_{Y_\sigma}) = \frac{rh(\widehat{Z}^l_\sigma)}{\widehat{Z}^l_\sigma} = \frac{\widehat{Z}^l_\sigma h(r)}{\widehat{Z}^l_\sigma} = \frac{h(r)}{1} = \overline{\eta}(h(r)) \upharpoonright_{Y_\sigma},$$

and therefore we have $ry = \overline{\eta}(h(r))$, hence

$$r\overline{\eta}'(y) = \overline{\eta}'(ry) = \overline{\eta}'(\overline{\eta}(h(r))) = \eta(h(r)) = \iota_l(rh) = r\iota_l(h) = rx$$

by a), and thus $r(\overline{\eta}'(y) - x) = 0$. This shows $I^l(\overline{\eta}'(y) - x) = 0$ and thus $\overline{\eta}'(y) - x \in {}^A\Gamma_I({}^AD_I(F))$. Since it holds ${}^A\Gamma_I({}^AD_I(F)) = 0$ by III.4.4.11 d) we get $\overline{\eta}'(y) = x$, and herewith the claim is proven. □

Now, the toric Serre-Grothendieck correspondence is at hand.

(4.2.9) Theorem *Suppose that $S_{(B)}$ has the ITR-property with respect to I_B.*

a) There exists a morphism $\zeta^0_{\Sigma,B} : \Gamma^{\Sigma,B}_{**} \to {}^BH^1_{I_B}$ of functors such that the sequence

$$0 \longrightarrow {}^B\Gamma_{I_B} \xrightarrow{\xi_{I_B}} \mathrm{Id}_{\mathsf{GrMod}^B(S_{(B)})} \xrightarrow{\overline{\eta}_{\Sigma,B}} \Gamma^{\Sigma,B}_{**} \xrightarrow{\zeta^0_{\Sigma,B}} {}^BH^1_{I_B} \longrightarrow 0$$

of functors is exact.

b) There exists a unique morphism of δ-functors

$$(\zeta^i_{\Sigma,B})_{i\in\mathbb{Z}} : \left(H^i_{**,\Sigma,B}\right)_{i\in\mathbb{Z}} \to \left({}^BH^{i+1}_{I_B}\right)_{i\in\mathbb{Z}},$$

and $\zeta^i_{\Sigma,B}$ is an isomorphism for every $i \in \mathbb{N}$.

Proof. This follows immediately from III.4.4.7 and 4.2.8. □

(4.2.10) The above Theorem 4.2.9 can be applied if R is Noetherian, for then the B-graded ring $S_{(B)}$ is Noetherian by III.3.4.7 and 2.1.6 b), and hence it has the ITR-property by III.4.2.4.

The last question treated is whether the components of the cohomology modules $H^i_{**,\Sigma,B}(F)$ (or of ${}^BH^i_{I_B}(F)$) are finitely generated or not. This is of interest for example to define cohomological Hilbert functions. Our result relies on a general theorem on proper morphisms of schemes and hence makes completeness of fans enter again.

(4.2.11) Proposition *Suppose that Σ is complete and that R is Noetherian. Let F be a finitely generated B-graded $S_{(B)}$-module, let $i \in \mathbb{Z}$, and let $\alpha \in B$. Then, the R-module $H^i_{**,\Sigma,B}(F)_\alpha$ is finitely generated.*

Proof. By 4.2.1 we have

$$H^i_{**,\Sigma,B}(F)_\alpha = H^i_{\alpha,\Sigma,B}(F) = H^i(X, \mathscr{S}_{\Sigma,B}(F(\alpha))).$$

Furthermore, since R is Noetherian, the same is true for $S_{(B)}$ by III.3.4.7 and 2.1.6 b), and since the B-graded $S_{(B)}$-module $F(\alpha)$ is finitely generated, it therefore follows from 3.1.9 b) that the $\mathscr{O}_{Y_\Sigma}$-module $\mathscr{S}_{\Sigma,B}(F(\alpha))$ is coherent.

Finally, since Σ is complete it is in particular full, and hence $Y_\Sigma(R)$ is proper over R by 2.3.3 and 1.3.12. Thus, the S_0-module $H^i(X, \mathscr{S}_{\Sigma,B}(F(\alpha)))$ is finitely generated by [ÉGA, III.3.2.3], and as $R = S_0$ by 2.1.7 the claim is proven. □

(4.2.12) Corollary *Suppose that Σ is complete and that R is Noetherian. Let F be a finitely generated B-graded $S_{(B)}$-module, let $i \in \mathbb{Z}$, and let $\alpha \in B$. Then, the R-module ${}^B H^i_{I_B}(F)_\alpha$ is finitely generated.*

Proof. Since Σ is complete it is in particular skeletal complete, and so we have $S_0 = R$ by 2.1.7. If $i < 0$, then the claim is obvious. If $i = 0$, then we have ${}^B H^0_{I_B}(F)_\alpha = {}^B\Gamma_{I_B}(F)_\alpha$. Since R is Noetherian and F is finitely generated it follows from III.3.4.7 and 2.1.6 b) that the B-graded $S_{(B)}$-module ${}^B\Gamma_{I_B}(F)$ is Noetherian, and therefore the S_0-module ${}^B\Gamma_{I_B}(F)_\alpha$ is Noetherian and hence finitely generated by III.3.3.6. If $i = 1$, then by 4.2.9 a) there is an epimorphism

$$\zeta^0_{\Sigma,B}(F)_\alpha : \Gamma^{\Sigma,B}_{**}(F)_\alpha \twoheadrightarrow {}^B H^1_{I_B}(F)_\alpha$$

in $\mathsf{Mod}(S_0)$, and so the claim follows from 4.2.11. Finally, for $i > 1$ the claim follows immediately from 4.2.9 b) and 4.2.11. □

(4.2.13) Suppose that R is Noetherian, and let F be a finitely generated A-graded S-module. If Σ is not complete, then the components ${}^A H^i_I(F)_\alpha$ of local cohomology, for $i \in \mathbb{Z}$ and $\alpha \in A$, are not necessarily finitely generated over R. As a counterexample we take for R a field and consider the $\mathbb{Z}^2$-fan Σ in $\mathbb{R}^2$ with maximal cones $\operatorname{cone}((1,0))$ and $\operatorname{cone}((0,1))$. Then, it holds $A = 0$, and S is the 0-graded polynomial algebra $R[Z_1, Z_2]$ over R. Moreover, we have $I = \langle Z_1, Z_2 \rangle_S$, but it follows from [2, 7.3.3] that the R-vector space $H^2_I(S)_0 = H^2_I(S) \cong H^2_{IS_I}(S)$ is not finitely generated.

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